ECE212 Cheat Sheet

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1 Circuit Analysis

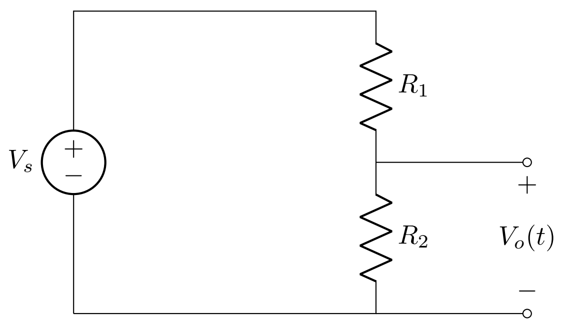
1.1 Ohm's Law

$$V = IR$$

1.2 Linearity Property

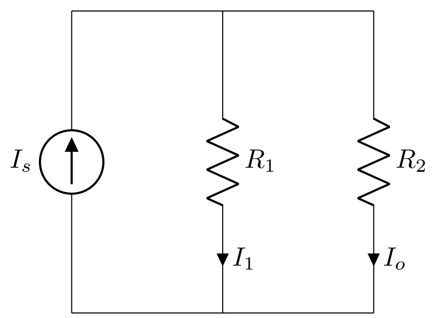
$$kiR = kv$$

1.3 Voltage Divider



$$V_o = \frac{R_2}{R_1 + R_2} V_s$$

1.4 Current Divider



$$I_o = \frac{R_1}{R_1 + R_2} I_s$$

2 Operational Amplifiers

2.1 Ideal OpAmp Characteristics

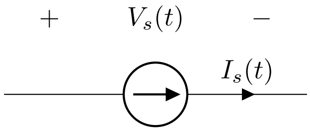
- $R_{in} \to \infty \Rightarrow i_N, i_P = 0$
- $R_{out} \to 0 \Rightarrow V_{out} = A(V_P - V_N)$
- $A \to \infty$

2.2 Ideal OpAmp Properties

- $V_P = V_N$
- $i_P = 0$
- $i_N = 0$
- Circuit is balanced ($R_{out} = 0$)

3 Element Constraints

3.1 Current Source



3.1.1 Time Domain

$$I_s(t) = I_m \cos(\omega t + \theta_s)$$

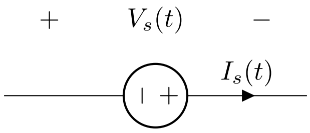
3.1.2 Frequency/Complex Domain

$$\bar{I}_s = I_m e^{j\theta_s}$$

3.1.3 S/Laplace Domain

$$I_s(s) = \mathscr{L}\{I_s(t)\}(s)$$

3.2 Voltage Source



3.2.1 Time Domain

$$V_s(t) = V_m \cos(\omega t + \theta_s)$$

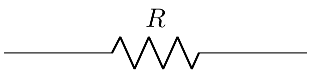
3.2.2 Frequency/Complex Domain

$$\bar{V_s} = V_m e^{j\theta_s}$$

3.2.3 S/Laplace Domain

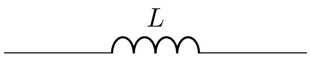
$$V_s(s) = \mathscr{L}\{V_s(t)\}(s)$$

3.3 Resistors



$$V = iR$$

3.4 Inductors



3.4.1 Time Domain

$$V_L = L\frac{di}{dt}$$

$$i_L = \frac{1}{L}\int_0^t V_L(t)dt + i_L(0)$$

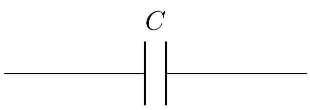
3.4.2 Frequency/Complex Domain

$$V_L(j\omega) = j\omega L I_L$$

3.4.3 S/Laplace Domain

$$V_L(s) = sLI_L$$

3.5 Capacitors



$$i_C = C\frac{dv}{dt}$$

$$V_C = \frac{1}{C}\int_0^t i_C(t)dt + i_C(0)$$

3.5.1 Frequency/Complex Domain

$$V_C(j\omega) = \frac{1}{j\omega C} I_C$$

3.5.2 S/Laplace Domain

$$V_C(s) = \frac{1}{sC} I_C$$

4 Coupled Inductors and Transformers

4.1 Flux

$$\phi(t) = kN_1 i(t)$$

4.2 Flux Linkage

$$\lambda = N_1\phi(t) = kN_1^2 i(t)$$

4.3 Voltage

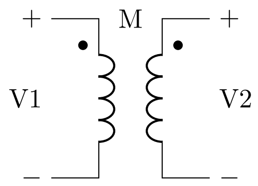
Voltage accross an inductor is given by

$$V(t) = \frac{d\lambda}{dt} = \frac{d}{dt}N_1\phi(t) = kN_1^2\frac{di}{dt}$$

Where

$$L = kN_1^2$$

4.4 Coupled Inductors



4.4.1 Time Domain

$$v_1(t) = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

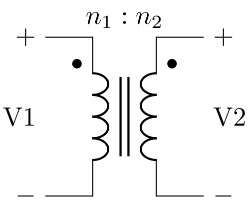
$$v_2(t) = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

4.4.2 Frequency/Complex Domain

$$\bar{V}_1 = j\omega L_1\bar{I}_1 + j\omega M\bar{I}_2$$

$$\bar{V}_2 = j\omega M\bar{I}_1 + j\omega L_2\bar{I}_2$$

4.5 Transformers



$$i_1 n_1 + i_2 n_2 + ... + i_k n_k = 0$$
$$\frac{V_1}{n_1} = \frac{V_2}{n_2} = ... = \frac{V_k}{n_k}$$

5 Complex Power

5.1 Complex Current/Voltage

$$v(t) = V_m \cos(\omega t + \theta_v)$$
$$i(t) = I_m \cos(\omega t + \theta_i)$$

$$\mathbf{V} = V_m \angle \theta_v \quad \mathbf{I} = I_m \angle \theta_i$$

5.2 Effective Value and RMS

The effective value of a periodic current is the dc current that delivers the same average power to a resistor as the periodic current

For a given periodic function $x(t)$, the RMS value is

$$X_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2 dt}$$

$$V_{rms} = \frac{V_m}{\sqrt{2}} \quad I_{rms} = \frac{I_m}{\sqrt{2}}$$

$$\mathbf{V}_{rms} = \frac{\mathbf{V}}{\sqrt{2}} = V_{rms}\angle\theta_v \quad \mathbf{I}_{rms} = \frac{\mathbf{I}}{\sqrt{2}} = I_{rms}\angle\theta_i$$

5.3 Instantaneous Power

$$p(t) = v(t)i(t)$$

$$p(t) = \frac{1}{2} V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i)$$

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$

5.4 Average Power

$$P = \frac{1}{T}\int_0^T p(t)dt$$

$$P = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)$$

$$P = \mathrm{Re}[\frac{1}{2}\mathbf{V}\mathbf{I}^*]$$

$$P = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i) = V_{rms}I_{rms}\cos(\theta_v - \theta_i)$$

5.5 Maximum Average Power Transfer

For a Thevenin equivalent circuit with impedance Z_{Th}

$$\mathbf{Z}_L = R_L + jX_L = R_{Th} - jX_{Th} = \mathbf{Z}_{Th}^*$$

5.6 Apparent Power

$$S = V_{rms}I_{rms}$$

$$S = |\mathbf{S}| = |\mathbf{V}_{rms}||\mathbf{I}_{rms}|$$

$$S = \sqrt{P^2 + Q^2}$$

5.7 Power Factor

$$\mathrm{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

- Unity Power Factor
  - $Q = 0$
  - Purely resistive loads
- Leading Power Factor
  - $Q < 0$
  - Capacitive loads
- Lagging Power Factor
  - $Q > 0$
  - Inductive loads

5.8 Complex Power

Units: VA

$$\mathbf{S} = P + Qj$$

$$\mathbf{S} = |\mathbf{V}_{rms}||\mathbf{I}_{rms}|\angle\theta_v - \theta_i$$

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \mathbf{V}_{rms}\mathbf{I}^*_{rms}$$

$$\mathbf{S} = I^2_{rms}\mathbf{Z} = \frac{V^2_{rms}}{\mathbf{Z}^*} = \mathbf{V}_{rms}\mathbf{I}^*_{rms}$$

5.9 Real Power

Also known as Active, or Average Power

Units: W

$$P = \mathrm{Re}(\mathbf{S}) = S\cos(\theta_v - \theta_i)$$

5.10 Reactive Power

Also known as Complex, or Imaginary Power

Units: VA_R

$$Q = \mathrm{Im}(\mathbf{S}) = S\sin(\theta_v - \theta_i)$$

6 Frequency Response

6.1 Transfer Functions

For a specific phasor input/output pair $X(s)$ and $Y(s)$, and $s = j\omega$

$$T(s) = \frac{Y(s)}{X(s)} = \frac{\mathscr{L}\{y(t)\}(s)}{\mathscr{L}\{x(t)\}(s)}$$

$X(s)$ and $Y(s)$ are the laplace transforms of time domain signals, such as voltage or current

6.2 Transfer Function Examples

Voltage Gain	$H(s) = \frac{V_o(s)}{V_i(s)}$
Current Gain	$H(s) = \frac{I_o(s)}{I_i(s)}$
Transfer Impedance	$H(s) = \frac{V_o(s)}{I_i(s)}$
Transfer Admittance	$H(s) = \frac{I_o(s)}{V_i(s)}$

6.3 Gain

Gain is measured in **decibels**

$$G_{dB} = 10\log_{10}\left(\frac{P_1}{P_2}\right)$$

Assuming $R_1 = R_2$ and using $P = IV = \frac{V^2}{R} = I^2R$

$$G_{dB} = 20\log_{10}\left(\frac{V_1}{V_2}\right)$$

$$G_{dB} = 20\log_{10}\left(\frac{I_1}{I_2}\right)$$

6.4 Bode Plot Features

Factor	Function	Gain	Phase
Constant	K	$20\log_{10}(K)$	$0°$ if $k > 0$, else $180°$
Zero at Origin	$j\omega$	$20\log_{10}(\omega)$	$90°$
Pole at Origin	$\frac{1}{j\omega}$	$-20\log_{10}(\omega)$	$-90°$
Zero at Origin of Order N	$(j\omega)^N$	$20N\log_{10}(\omega)$	$90N°$
Pole at Origin of Order N	$\frac{1}{(j\omega)^N}$	$-20N\log_{10}(\omega)$	$-90N°$
Simple Zero	$(1 + j\omega/z_1)$	$20\log_{10}\|1 + j\omega/z_1\|$	$\tan^{-1}(\frac{\omega}{z_1})$
Simple Pole	$\frac{1}{1+j\omega/z_1}$	$-20\log_{10}\|1 + j\omega/z_1\|$	-$\tan^{-1}(\frac{\omega}{z_1})$
Zero of Order N	$(1 + j\omega/z_1)^N$	$20N\log_{10}\|1 + j\omega/z_1\|$	$N\tan^{-1}(\frac{\omega}{z_1})$
Pole of Order N	$\frac{1}{1+(j\omega/z_1)^N}$	$-20N\log_{10}\|1 + j\omega/z_1\|$	-$N\tan^{-1}(\frac{\omega}{z_1})$

7 Trig Identities

7.1 Trigonometric Functions

For any complex number $z = x + iy$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\frac{d}{dz}\sin(z) = \cos(z) \qquad \frac{d}{dz}\cos(z) = -\sin(z)$$

7.2 Trig Identities

$$\left| \begin{array}{c|c} \tan(x) = \frac{\sin(x)}{\cos(x)} & \csc(x) = \frac{1}{\sin(x)} \\ \sec(x) = \frac{1}{\cos(x)} & \cot(x) = \frac{\cos(x)}{\sin(x)} \end{array} \right|$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$
$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$
$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$
$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

7.3 Hyperbolic Trig Functions

For any complex number $z = x + iy$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dz}\sinh(z) = \cosh(z) \qquad \frac{d}{dz}\cosh(z) = \sinh(z)$$

7.4 Hyperbolic Trig Identities

$$\left| \begin{array}{c|c} \tanh(z) = \frac{\sinh(z)}{\cosh(z)} & \operatorname{csch}(z) = \frac{1}{\sinh(z)} \\ \operatorname{sech}(z) = \frac{1}{\cosh(z)} & \coth(z) = \frac{\cosh(z)}{\sinh(z)} \end{array} \right|$$

$$\sinh(-z) = -\sinh(z)$$
$$\cosh(-z) = \cosh(z)$$
$$\cosh^2(z) - \sinh^2(z) = 1$$
$$1 - \tanh^2(z) = \operatorname{sech}^2(z)$$
$$\sinh(x + y) = \sinh(x)\cosh(x) + \cosh(y)\sinh(y)$$
$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(y)\sinh(y)$$
$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$
$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

8 Differential Equations

8.1 Solution Types for Homogenous 2nd Degree Linear ODE's

For an ODE of the form:

$$ay'' + by' + c = 0$$

and characteristic/auxiliary polynomial:

$$ar^2 + br + c = 0$$

with roots

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

8.1.1 Distinct Roots - Overdamped

$$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

8.1.2 Repeated Roots - Critically Damped

e.g. $r_1 = r_2 = r$

$$y_h = c_1 e^{rt} + c_2 t e^{rt}$$

8.1.3 Complex Conjugate Roots - Underdamped

If $r1, r2$ complex, then we can write $r1 = a + ib, r2 = a - ib$

$$y_h = e^{ax}(c_1 \cos(bx) + c_2 \sin(bx))$$

9 Laplace Transform

Let f be a function defined for $t \geq 0$

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

9.1 Basic Laplace Transforms

$\mathscr{L}\{1\} = \frac{1}{s}$	$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$
$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$	$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$
$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$	$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$

9.2 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\}(s) = \mathscr{L}\{f(t)\}(s-a) = F(s-a)$$

9.3 Second Translation Theorem

The **unit step function** $\mathrm{u}(t-a)$ is defined as

$$\mathrm{u}(t-a) = \begin{cases} 0 & \text{if } 0 \leq t < a \\ 1 & \text{if } t \geq a \end{cases}$$

If $\mathscr{L}\{f(t)\} = F(s)$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathrm{u}(t-a)\} = e^{-as}F(s)$$

9.4 Transforms of Derivatives

If $f, f', ...f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order, and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0)$$

where $F(s) = \mathscr{L}\{f(t)\}$

$$\mathscr{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

9.5 Derivatives of Transforms

If $\mathscr{L}\{f(t)\} = F(s)$ and $n = 1, 2, 3, ...$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

9.6 Transform of Integrals

$$\mathscr{L}\{\int_0^t f(\tau)d\tau\} = \frac{F(s)}{s}$$

$$\int_0^t f(\tau)d\tau = \mathscr{L}^{-1}\{\frac{F(s)}{s}\}$$

9.7 Convolution

9.7.1 Convolution Operation

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$, then the **convolution** of f and g, denoted by the symbol $f * g$, is

$$f * g = \int_0^t f(\tau) g(t - \tau) d\tau$$

9.7.2 Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\} \mathscr{L}\{g(t)\} = F(s)G(s)$$

9.7.3 Convolution in Inverse

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g$$

9.8 Dirac Delta Function

9.8.1 Unit Impulse

$$\delta_a(t - t_0) = \begin{cases} 0 & \text{if } 0 \leq t < t_0 - a \\ \frac{1}{2a} & \text{if } t_0 + a \leq t \leq t_0 + a \\ 0 & \text{if } t \geq t_0 + a \end{cases}$$

9.8.2 Dirac Delta Definition

$$\delta(t - t_0) = \begin{cases} \infty & \text{if } t = t_0 \\ 0 & \text{if } t \neq t_0 \end{cases}$$

Area under the distribution is 1:

$$\int_{-\infty}^{\infty} \delta(x - a) dx = 1$$

Sampling/Shifting Property:

$$\int_{-\infty}^{\infty} f(x)\delta(x - a) dx = f(a)$$

9.8.3 Dirac Delta Transform

For $t_0 > 0$

$$\mathscr{L}\{\delta(t - t_0)\} = e^{-st_0}$$

$$\mathscr{L}\{\delta(t)\} = 1$$

9.9 Additional Laplace Properties

9.9.1 Transform of a Periodic Function

If $f(t)$ is

- is piecewise continuous on $[0, \infty)$
- of exponential order
- periodic with period T

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$$

9.9.2 Volterra Integral Equation

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)d\tau$$

Taking Laplace transform of both sides:

$$F(s) = G(s) + \mathscr{L}\{f(t) * h(t)\} = G(s) + F(s)H(s)$$

9.9.3 Integrodifferential Equation

Circuit Analogy:

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$

10 Inverse Laplace Transforms

10.1 Basic Inverse Laplace Transforms

$\mathscr{L}^{-1}\{\frac{1}{s}\} = 1$	$\mathscr{L}^{-1}\{1\} = \delta(t)$
$\mathscr{L}\{\frac{1}{s-a}\} = e^{at}$	$\mathscr{L}^{-1}\{\frac{n!}{s^{n+1}}\} = t^n$
$\mathscr{L}^{-1}\{\frac{k}{s^2+k^2}\} = \sin(kt)$	$\mathscr{L}^{-1}\{\frac{s}{s^2+k^2}\} = \cos(kt)$
$\mathscr{L}^{-1}\{\frac{k}{s^2-k^2}\} = \sinh(kt)$	$\mathscr{L}^{-1}\{\frac{s}{s^2-k^2}\} = \cosh(kt)$