ECE216 Course Notes

Stephen Yang

January 2021

1 Complex Numbers

1.1 Euler's Formula

$$e^{jt} = \cos(t) + j\sin(t)$$

2 Linear Algebra

2.1 Linearly Independent

A **square** matrix B is invertible if

$$\det(B) \neq 0$$

3 Dirac Delta Distribution

$$\delta(x-a) = \begin{cases} 0 & \text{for } x \neq a \\ \infty & \text{for } x = a \end{cases}$$

3.1 Unit Impulse Properties

$$\delta(\alpha t) = \frac{1}{\alpha}\delta(t)$$

$$\delta(t) = \delta(-t)$$

$$\delta(t) = \frac{d}{dt}u(t)$$

3.2 Dirac Delta Integrals

Area under the distribution is 1:

$$\int_{-\infty}^{\infty} \delta(x-a)dx = 1$$

$$\int_{-\infty}^{a} \delta(x-a)dx = \int_{a}^{\infty} \delta(x-a)dx = \frac{1}{2}$$

Sampling/Shifting Property:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

4 Vector Spaces

4.1 $\mathbb{R}^N$ Inner Product

$$\langle x, y\rangle = \sum_{n=0}^{N-1} x[n]y[n]$$

4.1.1 Properties

Symmetry:

$$\langle x, y\rangle = \langle y, x\rangle$$

Self product:

$$\langle x, x\rangle = \sum_{n=0}^{N-1} (x[n])^2 = ||x||^2$$

Distributive:

$$\langle x, \alpha_1 y + \alpha_2 z\rangle = \alpha_1 \langle x, y\rangle + \alpha_2 \langle x, y\rangle$$

4.1.2 Projection Z_{opt}

$$\alpha_{opt} = \frac{\langle x, y\rangle}{\langle y, y\rangle} = \frac{\langle x, y\rangle}{||y||^2}$$

$$\hat{z} = \alpha_{opt}\hat{y} = \frac{\langle x, y\rangle}{\langle y, y\rangle}\hat{y}$$

4.1.3 2-Norm

$$||x|| = \sqrt{\sum_{n=0}^{N-1} (x[n])^2}$$

$$||x||^2 = \sum_{n=0}^{N-1} (x[n])^2$$

$$||x-y|| = \sqrt{\sum_{n=0}^{N-1} (x[n]-y[n])^2}$$

4.1.4 Angle

$$\cos(\theta) = \frac{||z_{opt}||}{||x||} = \frac{|\langle x, y \rangle|}{||x||\,||y||}$$

4.2 $\mathbb{C}^N$ Inner Product

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x[n]y[n]^*$$

4.2.1 Properties

Conjugate Symmetry:

$$\langle x, y \rangle = \langle y, x \rangle^*$$

Self product:

$$\langle x, x \rangle = \sum_{n=0}^{N-1} x[n]x[n]^* = ||x||^2$$

Conjugate Distributive:

$$\langle \alpha_1 x + \alpha_2 y, z \rangle = \alpha_1 \langle x, z \rangle + \alpha_2 \langle y, z \rangle$$
$$\langle x, \alpha_1 y + \alpha_2 z \rangle = \alpha_1^* \langle x, y \rangle + \alpha_2^* \langle x, y \rangle$$

4.2.2 Projection Z_{opt}

$$\alpha_{opt} = \frac{\langle x, y \rangle}{\langle y, y \rangle} = \frac{\langle x, y \rangle}{||y||^2}$$

$$\hat{z} = \alpha_{opt}\hat{y} = \frac{\langle x, y \rangle}{\langle y, y \rangle}\hat{y}$$

4.2.3 2-Norm

$$||x|| = \sqrt{\sum_{n=0}^{N-1} (x[n]^* x[n])} = \sqrt{\sum_{n=0}^{N-1} (|x[n]|)^2}$$

$$||x||^2 = \sum_{n=0}^{N-1} (x[n]^* x[n])$$

$$||x - y|| = \sqrt{\sum_{n=0}^{N-1} (x[n] - y[n])^* (x[n] - y[n])}$$

4.2.4 Angle

$$\cos(\theta) = \frac{||z_{opt}||}{||x||} = \frac{|\langle x, y \rangle|}{||x||\,||y||}$$

4.3 Cauchy-Schwarz Inequality

$$0 \leq \frac{|\langle x, y \rangle|}{||x|| \, ||y||} = \cos\theta \leq 1$$

5 Series

5.1 Arithmetic Series

$$\sum_{n=0}^{N} a = \frac{1}{2} n(n+1)$$

5.2 Geometric Series

$$\sum_{n=0}^{N} a^n = \begin{cases} \frac{1-a^{N+1}}{1-a} & a \neq 1 \\ N & a = 1 \end{cases}$$

For $-1 < r < 1$

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

6 Continuous Time Signals

6.1 Frequency and Period

$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f}$$

$$T_0 \omega_0 = 2\pi$$

6.2 Even Component

$$x_{even}(t) = \frac{x(t) + x(-t)}{2}$$

6.3 Odd Component

$$x_{odd}(t) = \frac{x(t) - x(-t)}{2}$$

6.4 Energy

$$E = \int_{t_1}^{t_2} |x(t)|^2 dt$$

6.5 Average Power

$$P_{avg} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt$$

6.6 Synthesis

Approximation:

$$\hat{x}_K(t) = \sum_{k=-K}^{K} a_k e^{jk\omega_0 t}$$

Exact (Continuous Time Fourier Series):

$$x(t) = \hat{x}_\infty(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

6.7 Analysis

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

6.8 Parseval's Relation

$$P_{avg} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

7 Discrete Time Signals

7.1 Frequency and Period

A DT signal is periodic if

$$x[n] = x[n+N]$$

For a signal $x(t)$ to be periodic,

$$\nu_k N_0 = 2\pi k \quad \text{for some integer } k$$

$$\omega_0 = \frac{2\pi}{N_0}$$

$$\nu_k = w_o k$$

7.1.1 Constant DT Signal

Any **constant** DT signal is periodic for all $N \in \mathbb{Z}$, with

$$N_0 = 1$$

7.1.2 Complex Exponential DT

For a complex exponential DT signal (e.g. $x[n] = e^{j\omega_0 n}$) to be periodic,

$$\frac{\omega_0}{2\pi} \in \mathbb{Q}$$

($\frac{\omega_0}{2\pi}$ must be rational)

$$\omega_0 N_0 = 2\pi k, \quad k \in \mathbb{Z}$$

And $e^{j\omega_0 n}$ is **always** a 2π periodic function of frequency

7.1.3 Aliasing

Different CT signals can produce the exact same DT signal when sampled

$$x_{DT}[n] = x_{CT}(t)|_{t=T_s}$$

$$\cos((2\pi/8)n) = \cos((2\pi/8 + 2\pi)n) = \cos((2\pi/8 + 4\pi)n)$$

7.2 Even Component

$$x_{even}[n] = \frac{x[n] + x[-n]}{2}$$

7.3 Odd Component

$$x_{odd}[n] = \frac{x[n] - x[-n]}{2}$$

7.4 Energy

$$E = \sum_{n=n_1}^{n_2} |[n]|^2$$

7.5 Average Power

$$P_{avg} = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |[n]|^2$$

7.6 Special Signals

7.6.1 DT Unit Step

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

7.6.2 DT Unit Impulse

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

7.7 DT Basis

$$\phi_k[n] = e^{j\nu_k n} = e^{j\omega_0 kn}$$

7.8 DT Orthogonality Principle

$$\frac{1}{N_0}\sum_{k=0}^{N_0-1} e^{j\omega_0 k(n-l)} = \begin{cases} 1 & n = l \\ 0 & n \neq l \end{cases}$$

$$\sum_{k=0}^{N_0-1} e^{j\omega_0 k(n-l)} = \begin{cases} N_0 & n = l \\ 0 & n \neq l \end{cases}$$

7.9 Synthesis

Approximation:

$$\hat{x}_K\left[n\right] = \sum_{k=0}^{K-1} \alpha_k e^{\mathbf{j}w_o kn} = \sum_{k=0}^{K-1} \alpha_k \phi_k[n]$$

Exact (Discrete Time Fourier Series):

$$x\left[n\right] = \hat{x}_{N_0}\left[n\right] = \sum_{k=0}^{N_0-1} \alpha_k e^{\mathbf{j}w_o kn}$$

7.10 Analysis

$$\alpha_k = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x\left[n\right] e^{-\mathbf{j}w_o kn}$$

7.11 H Matrix

The matrix H (also called the Discrete Fourier Transform Matrix) is symmetric and has complex elements

$$H_{kn} = e^{-j\omega_0 kn}$$

$$H = \begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & e^{-j\omega_0} & e^{-j\omega_0(2)} & ... & e^{-j\omega_0(N_0-1)} \\ 1 & e^{-j\omega_0(2)} & e^{-j\omega_0(4)} & ... & e^{-j\omega_0(2)(N_0-1)} \\ ... & ... & ... & ... & ... \\ 1 & e^{-j\omega_0(N_0-1)} & e^{-j\omega_0(2)(N_0-1)} & ... & e^{-j\omega_0(N_0-1)(N_0-1)} \end{bmatrix}$$

7.12 G Matrix

$$G_{kn} = e^{j\omega_0 kn}$$

$$G = \begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & e^{j\omega_0} & e^{j\omega_0(2)} & ... & e^{j\omega_0(N_0-1)} \\ 1 & e^{j\omega_0(2)} & e^{j\omega_0(4)} & ... & e^{j\omega_0(2)(N_0-1)} \\ ... & ... & ... & ... & ... \\ 1 & e^{j\omega_0(N_0-1)} & e^{j\omega_0(2)(N_0-1)} & ... & e^{j\omega_0(N_0-1)(N_0-1)} \end{bmatrix}$$

7.13 Relationship Between H and G

$$\frac{1}{N_0}\mathbf{GH} = \mathbf{I}$$

$$HH^H = HG = N_0 I$$

Where I is the Identity Matrix

$$H_{kn} = G^*_{nk}$$

Additionally, H and G are Hermitian conjugates

$$G = H^H$$

7.14 Matrix Vector Notation for DTFS

$$x = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ ... \\ x[N_0-1] \end{bmatrix}$$

$$\alpha = \frac{1}{N_0}Hx$$

$$\hat{x}_{N_0} = G\alpha = \frac{1}{N_0}GHx$$

7.15 Numerical Approximation for CTFS with DTFS

7.15.1 CTFS definition

$$\alpha_{ct,k} = \frac{1}{T_0}\int_0^{T_0} x_{ct}(t)e^{-j\omega_0 kt}dt, \quad \omega_0 = \frac{2\pi}{T_0}$$

7.15.2 DTFS sampling of CTFS

$$x_{ct}(t)|_{t=nT_s} = x_{ct}(nT_s) = x[n]$$

To ensure $x[n]$ periodic, need to sample **integer** number of times per period T_0:

$$T_s = \frac{T_0}{N_0}$$

where

- N_0 is the number of samples per period
- T_S is the sampling rate

7.15.3 CTFS approximation

$$\alpha_{ct,k} \approx \frac{1}{T_0} \sum_{n=0}^{N_0+1} x_{ct}(nT_s) T_s e^{-j\omega_0 k(nT_s)}$$

$$\alpha_{ct,k} \approx \frac{T_s}{T_0} \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi}{T_0}k(nT_s)} = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x\left[n\right] e^{-\mathbf{j}w_o kn} = \alpha_{dt,k}$$

8 Geometric DTFS

$$\alpha = (\alpha_0, \alpha_1, ..., \alpha_{N_0-1})$$

$$x = (x[0], x[1], ..., x[N_0 - 1])$$

$$\hat{\phi_k} = (1, e^{j\omega_0 k(1)}, e^{j\omega_0 k(2)}, ..., e^{j\omega_0 k(N_0-1)})$$

8.1 Orthogonal Basis

$$\phi_k[n] = e^{j\omega_0 kn}, \quad n \in \{0, ..., N_0 - 1\}$$

8.2 Orthogonality Principle

$$\langle \phi_k, \phi_j \rangle = \sum_{n=0}^{N_0-1} \phi_k[n] \phi_j[n]^* = \begin{cases} N_0 & k = j \\ 0 & k \neq j \end{cases}$$

These N_0 vectors form an orthogonal basis for $l_2([0, N_0 - 1])$

8.3 Analysis

$$\alpha_k = \frac{\langle x, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} = \frac{1}{N_0} \hat{x}^T \hat{\phi_k}^* = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] \phi_k[n]^* = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j\omega_0 kn}$$

9 Discrete Time Systems

9.1 Notation

$T\{x\}$ is the *entire* output signal

$T\{x\}[n]$ is the value of the output signal at time $n \in \mathbb{Z}$

9.2 Finite Impulse Response Systems

$$y = T\{x\} = \sum_{k=0}^{M} b_k x_k$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + ... + b_M x[n-M]$$

Where M is the order of the FIR filter

9.2.1 Impulse Response Case

$$x[n] = \delta[n]$$

$$y[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n & 0 \le n \le M \\ 0 & otherwise \end{cases}$$

9.3 Memoryless

A DT system T is memoryless if $y[n]$ only depends on time instance $x[n]$

- $y[n] = x[n]$ is memoryless
- $y[n] = (x[n])^2$ is memoryless
- $y[n] = x[n-1]$ is **not** memoryless

Additionally, if

$$h[n] = 0 \quad \forall n \neq 0$$

Then the system is memoryless

9.4 Causality

A DT system T is causal if $y[n]$ only depends on present and past inputs (not future inputs).

- $y[n] = x[n]$ is causal
- $y[n] = x[n-1]$ is causal
- $y[n] = x[n+1]$ is **not** causal

Additionally, if

$$h[n] = 0 \quad \forall n < 0$$

Then the system is causal

9.5 Stability

9.5.1 Bounded Signal

A DT signal is bounded if there exists $A > 0$ for

$$x[n] \leq A \quad \forall n \in \mathbb{Z}$$

9.5.2 BIBO Stability

A DT system T is **Bounded-Input Bounded-Output** (BIBO) Stable if for any **bounded** input x, the output $y = T\{x\}$ is also bounded

9.5.3 Absolutely Summable

A DT LTI system is BIBO Stable if and only if $h[n]$ is **absolutely summable**

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

9.6 Invertibility

A DT system T is invertible if there exists another DT system T_{inv} such that

$$T_{inv}\{T\{x\}\} = T\{T_{inv}\{x\}\} = x$$

for all x

9.7 LTI Systems

9.7.1 Time-Invariance

A DT system T is time-invariant if for any input x and output signal $y = T\{x\}$,

$$y_k = y[n-k] = T\{x\}$$

9.7.2 Linearity

A DT system T is linear if for any input signals x^1, x^2

$$T\{\alpha_1 x^1 + \alpha_2 x^2\} = \alpha_1 T\{x^1\} + \alpha_2 T\{x^2\}$$

Or in general (**Superposition Principle**)

$$T\left\{\sum_{j=1}^{n} \alpha_j x^j\right\} = \sum_{j=1}^{n} \alpha_j T\{x^j\}$$

9.8 Pulse Train Decomposition

9.8.1 Basis of Vector Space for all DT Signals

$$\phi_k = \delta_k$$

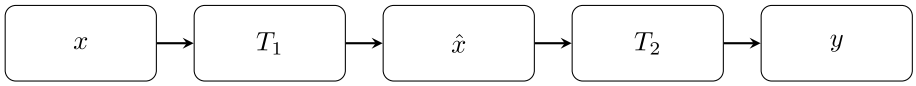
9.8.2 Synthesis

$$x = \sum_{k=-\infty}^{\infty} \frac{\langle x, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \phi_k = \sum_{k=-\infty}^{\infty} x[k] \delta_k$$

9.8.3 Analysis

$$\alpha_k = \frac{1}{\langle \phi_k, \phi_k \rangle} \langle x, \phi_k \rangle = x[k]$$

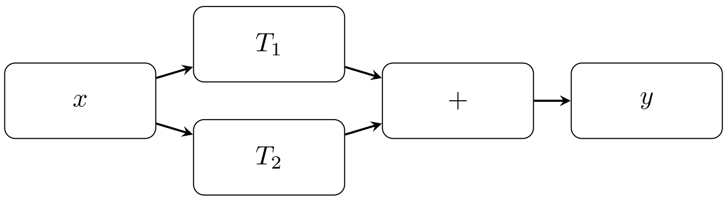
9.9 Series Connection of LTI Systems



$$y = h_2 * \hat{x} = h_2 * (h_1 * x) = (h_2 * h_1) * x$$

$$h = h_2 * h_1 = h_1 * h_2$$

9.10 Parallel Connection of LTI Systems



$$y = T_1\{x\} + T_2\{x\} = h_1 * x + h_2 * x = (h_1 + h_2) * x$$

$$h = (h_1 + h_2)$$

10 Convolution

$$x * y = \sum_{k=-\infty}^{\infty} x[n-k] \cdot y[k]$$

$$x * y = \int_{-\infty}^{\infty} x_\tau(t) y(\tau) d\tau = \int_{-\infty}^{\infty} x(t-\tau) y(\tau) d\tau$$

10.1 Commutative Property

$$x * y = y * x = \sum_{k=-\infty}^{\infty} x[k] \cdot y[n-k]$$

$$x * y = y * x = \int_{-\infty}^{\infty} x_\tau(t) y(\tau) d\tau = \int_{-\infty}^{\infty} y_\tau(t) x(\tau) d\tau$$

10.2 Associative Property

$$(x * y) * z = x * (y * z)$$

10.3 Linearity Property

$$x * (\alpha y + \beta z) = x * \alpha y + x * \beta z$$

10.4 Basic Convolutions

10.4.1 Unit Impulse

10.4.2 Unit Step

11 Impulse Response

The **impulse response** h of T is defined to be the output of T in response to an impulse input δ_0 at time $n = 0$

$$h = T\{\delta_0\}$$

$$y = f(x) \quad \longrightarrow \quad h = f(\delta)$$

11.1 Impulse Convolution

For any signal x

$$y = h * x$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] = \sum_{l=-\infty}^{\infty} h[l]x[n-l]$$

$$y(t) = \int_{-\infty}^{\infty} h_\tau(t)x(\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

12 Discrete-Time Systems in Frequency

12.1 DT Complex Exponential Signal

$$x[n] = Ae^{j\omega n+\theta} = Ae^{j\theta}e^{j\omega n}$$

12.2 Frequency Response

$$H(e^{j\omega}) \equiv \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

For LTI Systems, and with a input **Complex Exponential DT Signal** $x[n] = Ae^{j\theta}e^{j\omega n}$,

$$y = T\{x\} = \left[\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right] x$$

$$y = T\{x\} = H(e^{j\omega})x$$

12.2.1 Frequency Response Notation

Usually, frequency response is written in polar form,

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}$$

$$y = H(e^{j\omega})x = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})}Ae^{j\theta}e^{j\omega n}$$

$$y = A|H(e^{j\omega})|e^{j(\theta+\angle H(e^{j\omega}))}e^{j\omega n}$$

12.2.2 LTI System Observations

12.2.3 Eigenfunction Property

An LTI system produces a DT exponential output (with the same frequency) as the input DT exponential

12.2.4 Amplitude Scaling

An LTI system scales the amplitude of the input signal by $|H(e^{j\omega})|$

12.2.5 Phase Shifting

An LTI system shifts the phase of the input signal by $\angle H(e^{j\omega})$ radians

12.2.6 Periodicity

The frequency response $H(e^{j\omega})$ has a period

$$T_\omega = 2\pi$$

Or,

$$H(e^{j(\omega+2\pi l)}) = H(e^{j\omega})$$

12.3 Conjugate Symmetry

If the impulse response h of a DT LTI system is real-valued, then the frequency response H is **conjugate symmetric**,

$$H(e^{j\omega})^* = H(e^{-j\omega}) \quad \forall\, \omega \in \mathbb{R}$$

$|H(e^{j\omega})|$ is an **even function** of $\omega \in \mathbb{R}$
$\angle H(e^{j\omega})$ is an **odd function** of $\omega \in \mathbb{R}$

13 DT Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

13.1 Condition for Convergence

If x is absolutely summable, then the DTFT exists

$$|X(e^{j\omega})| \leq \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| = \left| \sum_{n=-\infty}^{\infty} x[n] \right| < \infty$$

13.2 Inverse DT Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

13.3 Frequency Response and DTFT

$$Y(e^{j\omega}) = \frac{H(e^{j\omega})}{X(e^{j\omega})} \quad \longrightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

13.4 Common DTFT's

Time Domain $x[n]$	Frequency Domain $X(e^{j\omega})$
$\delta[n]$	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$
$e^{j\nu n}$	$2\pi\delta(\omega - \nu)$
$u[n]$	$\frac{1}{1-e^{-j\omega}}$

13.5 Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

14 Continuous Time Systems

14.1 Memoryless

A CT system T is memoryless if for all $t \in \mathbb{R}$, the output value $y(t)$ depends only on $x(t)$

- $y(t) = (x(t))^2$ is memoryless
- $y(t) = x(t - \tau)$ is not memoryless for $\tau \neq 0$

Additionally, if

$$h(t) = \alpha\delta(t) \quad \alpha \in \mathbb{R}$$

Then the system is memoryless

14.2 Causality

A CT system T is causal if for all $t \in \mathbb{R}$, the output value $y(t)$ depends only on the present and previous values $\{x(\tau)\}_{\tau \leq t}$

- $y(t) = x(t - \tau)$ is causal for $\tau \geq 0$
- $y(t) = x(t + 1)$ is not causal

Additionally, if

$$h(t) = 0 \quad \forall t < 0$$

Then the system is causal

14.3 Stability

14.3.1 Bounded Signal

A CT signal is bounded if there exists A¿0 for

$$x(t) \leq A \quad \forall t \in \mathbb{Z}$$

14.3.2 BIBO Stability

A CT system is **Bounded-Input Bounded-Output** (BIBO) stable if for any **bounded** input x, the corresponding output $y = T\{x\}$ is also bounded.

14.3.3 Absolutely Summable

A CT LTI system is BIBO Stable if and only if $h(t)$ is **absolutely integrable**

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

14.4 Invertibility

A CT system is invertible if there exists another CT system T_{inv} such that $T_{inv}\{T\{x\}\} = T\{T_{inv}\{x\}\} = x$ for all input signals x

14.5 LTI Systems

14.5.1 Time-Invariance

A CT system T is time-invariant if for any input signal x with output signal $y = T\{x\}$, it holds that $y_\tau = T\{x_\tau\}$ for all possible time shifts $\tau \in \mathbb{R}$

14.5.2 Linearity

A CT system T is linear if for any input signals x^1, x^2

$$T\{\alpha_1 x^1 + \alpha_2 x^2\} = \alpha_1 T\{x^1\} + \alpha_2 T\{x^2\}$$

Or in general (**Superposition Principle**)

$$T\left\{\sum_{j=1}^{n} \alpha_j x^j\right\} = \sum_{j=1}^{n} \alpha_j T\{x^j\}$$

15 Continuous-Time Systems in Frequency

15.1 CT Complex Exponential Signal

$$x(t) = Ae^{j(\omega t+\theta)} = Ae^{(j\theta)}e^{(j\omega t)}$$

15.2 Frequency Response

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

For LTI Systems, and with an input **Complex Exponential CT Signal** $x(t) = Ae^{(j\theta)}e^{(j\omega t)}$

$$y = T\{x\} = \left[\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right] x$$

$$y = T\{x\} = H(j\omega)x$$

15.3 Frequency Response Notation

Usually, frequency response is written in polar form,

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

$$y = H(j\omega)x = |H(j\omega)|e^{j\angle H(j\omega)}Ae^{j\theta}e^{j\omega t}$$

$$y = A|H(j\omega)|e^{j(\theta+\angle H(j\omega))}e^{j\omega t}$$

15.4 LTI System Observations

15.4.1 Eigenfunction Property

An LTI system produces a CT exponential output (with the same frequency) as the input CT exponential

15.4.2 Amplitude Scaling

An LTI system scales the amplitude of the input signal by $|H(j\omega)|$

15.4.3 Phase Shifting

An LTI system shifts the phase of the input signal by $\angle H(j\omega)$ radians

15.5 Conjugate Symmetry

If the impulse response h of a CT LTI system is real-valued, then the frequency response H is **conjugate symmetric**, meaning

$$H(j\omega)^* = H(-j\omega) \quad \forall \omega \in \mathbb{R}$$

$|H(j\omega)|$ is an **even** function of $\omega \in \mathbb{R}$
$\angle H(j\omega)$ is an **odd** function of $\omega \in \mathbb{R}$

16 CT Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

16.1 Condition for Convergence

If x is absolutely integrable, then the CTFT exists

$$|X(j\omega)| \leq \left| \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \right| = \left| \int_{-\infty}^{\infty} x(t)dt \right| < \infty$$

16.2 Inverse CT Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

16.3 Frequency Response and CTFT

$$Y(j\omega) = H(j\omega)X(j\omega)$$

16.4 Common CTFT's

Time Domain $x(t)$	Frequency Domain $X(j\omega)$
$\delta(t-\tau)$	$e^{-j\omega\tau}$
$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
$u(t+\tau/2)-u(t-\tau/2)$	$\frac{\sin(\omega\tau/2)}{\omega/2}$
$\frac{\sin(\omega_b t)}{\pi t}$	$u(\omega+\omega_b)-u(\omega-\omega_b)$
$e^{j\nu t}$	$2\pi\delta(\omega-\nu)$
$\frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{j\omega_0 kt}$	$\frac{2\pi}{T_0}\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0 k)$

16.5 Parseval's Relation

Let x be a CT signal with CTFT spectrum X. If x is square-integrable, then

$$\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega$$

17 Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

For $s = j\omega$, the Laplace Transform is exactly equal to the CTFT

17.1 Transfer Function

$$Y(s) = H(s)X(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

Where $H(s)$ is the **transfer function**

18 Sampling Theory

18.1 Sampling Function

$$s(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT_0)$$

Where $T_0 > 0$ is the Sampling Period

19 Circular Convolution

$$y = x \circledast h$$

$$y[n] = \sum_{l=0}^{N_0-1} h[n-l]x[l]$$

20 Trigonometry

20.1 Trigonometric Functions

For any complex number $z = x + iy$

$$\sin(z) = \frac{e^{jz} - e^{-jz}}{2i} \quad \cos(z) = \frac{e^{jz} + e^{-jz}}{2}$$

$$\frac{d}{dz}\sin(z) = \cos(z) \quad \frac{d}{dz}\cos(z) = -\sin(z)$$

20.2 Trig Identities

$$\begin{array}{|c|c|} \tan(x) = \frac{\sin(x)}{\cos(x)} & \csc(x) = \frac{1}{\sin(x)} \\ \sec(x) = \frac{1}{\cos(x)} & \cot(x) = \frac{\cos(x)}{\sin(x)} \end{array}$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$
$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$
$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$
$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$$

$$\cos(\alpha)\cos(\beta) = \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2}$$

$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$

20.3 Trig Integral Identities

$$\int_0^\infty e^{-\alpha t}\cos(bt)dt = \frac{a}{a^2 + b^2}$$
$$\int_0^\infty e^{-\alpha t}\sin(bt)dt = \frac{-b}{a^2 + b^2}$$

20.4 Hyperbolic Trig Functions

For any complex number $z = x + iy$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dz}\sinh(z) = \cosh(z) \qquad \frac{d}{dz}\cosh(z) = \sinh(z)$$

20.5 Hyperbolic Trig Identities

$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$	$\operatorname{csch}(z) = \frac{1}{\sinh(z)}$
$\operatorname{sech}(z) = \frac{1}{\cosh(z)}$	$\coth(z) = \frac{\cosh(z)}{\sinh(z)}$

$$\sinh(-z) = -\sinh(z)$$

$$\cosh(-z) = \cosh(z)$$

$$\cosh^2(z) - \sinh^2(z) = 1$$

$$1 - \tanh^2(z) = \operatorname{sech}^2(z)$$

$$\sinh(x + y) = \sinh(x)\cosh(x) + \cosh(y)\sinh(y)$$

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(y)\sinh(y)$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

20.6 Sinc Function

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}$$