1 Set Theory Review

1.1 Union and Intersection

$$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$$

$$A \cap B = \{x : x \in A \quad \text{and} \quad x \in B\}$$

1.2 Complement

$$A^C = \{x : x \notin A\}$$

1.3 Disjoint Sets

Two sets A_i and A_j are disjoint if

$$A_i \cap A_j = \varnothing \quad \forall i, j \; i \neq j$$

1.4 Collectively Exhaustive Sets

Sets $A_i, \ldots, A_n$ are collectively exhaustive if

$$\cup_{i=1}^{N} A_i = S$$

1.5 Partition

Sets $A_i, \ldots, A_n$ are called a partition of S if $A_i, \ldots, A_n$ are disjoint and collectively exhaustive.

1.6 Properties of Sets

1.6.1 Commutative

$$A \cap B = B \cap A \qquad A \cup B = B \cup A$$

1.6.2 Associative

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

1.6.3 Distributive

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

1.7 Relative Complement/Difference

$$A - B = \{x : x \in A \quad \text{and} \quad x \notin B\}$$

$$A - B = A \cap B^C$$

1.7.1 DeMorgan's

$$(A \cup B)^C = A^C \cap B^C \qquad (A \cap B)^C = A^C \cup B^C$$

2 Probability Theory Introduction

2.1 Relative Frequency

Suppose that an experiment is repeated n times under identical conditions. Let $N_0(n), N_1(n), \ldots, N_k(n)$ be the number of times the outcome k happens. Then the relative frequency of outcome k is

$$f_k(n) = \frac{N_k(n)}{n} \quad \text{where} \quad \lim_{n \to \infty} f_k(n) = p_k$$

2.2 Axioms of Probability

$$P(A) \geq 0 \qquad P[S] = 1$$

$$A \cap B = \varnothing \quad \longrightarrow \quad P(A \cup B) = P(A) + P(B)$$

$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P[A_k]$$

If A_1, A_2 is a sequence of events s.t. $A_i \cap A_j = \varnothing \quad i \neq j$

2.3 Bayesian Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

3 Counting Methods and Sampling

Permutations of n distinct objects (k-tuples):	$n!$
Number of **ordered** samples with size k **with** replacement:	n^k
Number of **ordered** samples with size k **without** replacement:	$\frac{n!}{(n-k)!}$
Number of **unordered** samples with size k **without** replacement:	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
Number of **unordered** samples with size k and **with** replacement:	$\binom{n-1+k}{k} = \binom{n-1+k}{n-1}$

3.1 Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \binom{n}{k} = \binom{n}{n-k}$$

4 Conditional Probability

If A and B are related, then the conditional probability of A *given* that B (and $P[B] > 0$) has occurred is

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

5 Theorem of Total Probability

For $B_1, B_2, \ldots, B_n$ mutually exclusive events whose union equals the sample space S (e.g. $B_1, \ldots, B_n$ is a **partition** of S), then

$$P[A] = P[A|B_1]P[B_1] \ldots + P[A|B_n]P[B_n]$$

6 Bayes Rule

For $B_1, B_2, \ldots, B_n$ a partition of sample space S,

$$P[B_j|A] = \frac{P[A \cap B_j]}{P[A]} = \frac{P[A|B_j]P[B_j]}{\sum_{k=1}^{n} P[A|B_k]P[B_k]}$$

7 Independence

If knowledge of the occurrence of event B does not alter the probability of event A, then event A is independent of B.

$$P[A] = P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Define A, B to be independent if

$$P[A \cap B] = P[A]P[B]$$

Then

$$P[A|B] = P[A], \; P[B|A] = P[B]$$

$$P[A^C \cap B^C] = P[A^C]P[B^C]$$

7.1 Notes on Independence

If two events have nonzero probability ($P[A] > 0, P[B] > 0$), and are mutually exclusive, then they cannot be independent

7.2 Triplet Independence

For three events A, B, C to be independent,

- A,B,C Pairwise Independent
- knowledge of occurrence of any two events (e.g. A, B) should not effect the prob of the third (C)

7.2.1 Pairwise Independence

$$P[A \cap B] = P[A]P[B] \quad P[A \cap C] = P[A]P[C]$$

$$P[B \cap C] = P[B]P[C]$$

7.2.2 Independence of Events

$$P[C|A \cap B] = \frac{P[A \cap B \cap C]}{P[A \cap B]} = P[C]$$

Finally, for **Triplet Independence**, we must have

$$P[A \cap B \cap C] = P[A]P[B]P[C]$$

8 Sequential Experiments

8.1 Bernoulli Trials

Let k be the num of successes in n independent Bernoulli trials. Then the probabilities of k are given by **binomial probability law**

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for} \quad k = 0, \ldots, n$$

8.2 Multinomial Probability Law

Let $B_1, B_2, \ldots, B_M$ be a partition of the sample space S, and let $P[B_j] = p_j$. Also, the events are disjoint:

$$p_1 + p_2 + \ldots + p_M = 1$$

The multinomial probability law is

$$P[(k_1, \ldots, k_M)] = \frac{n!}{k_1! \ldots k_M!} p_1^{k_1} \ldots p_M^{k_M}$$

8.3 Geometric Probability Law

The probability that more than K trials are required before a success (with probability $p, q = 1-p$) occurs in a series of repeated independent Bernoulli trials is

$$P[m > K] = p \sum_{m=K+1}^{\infty} q^{m-1} = pq^K \frac{1}{1-q} = q^K$$

The probability that K trials are required for a success (with probability $p, q = 1-p$) is

$$P[m = K] = (p)(1-p)^{(K-1)} = pq^{(K-1)}$$

8.3.1 Hypergeometric Distribution

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Where K is the number of success in the population, k is the number of observed successes, N is the population size, and n is the sample size.

9 Random Variables (RV)

A **Random Variable** X is a function that assigns a real number $X(\zeta)$ to each outcome ζ in the sample space of a random experiment.

10 Discrete Random Variable (DRV)

A **Discrete Random Variable X** is defined as a random variable that assumes values from a countable set.

10.1 PMF

$$p_X(x) = P[X = x] = P[\{\zeta : X(\zeta) = x\} \quad x \in \mathbb{R}$$

For x_k in S_X, $\quad p_X(x_k) = P[A_k]$

10.1.1 PMF Properties

$$p_X(x) \geq 0 \quad \forall x$$

$$\sum_{x \in S_X} p_X(x) = \sum_k p_X(x_k) = \sum_k P[A_k] = 1$$

$$P[X \text{in} B] = \sum_{x \in B} p_X(x) \quad \text{where } B \subset S_X$$

10.2 Conditional PMF

Let X be a DRV with PMF $P_X(x)$, and $\exists C, P[C] > 0$. The **Conditional PMF** is given by

$$p_X(x|C) = P[X = x|C] = \frac{P[\{X = x\} \cap C]}{P[C]}$$

10.3 Expected Value

The **expected value** (or **mean**) of a DRV is

$$E[X] = \sum_{x \in S_X} x p_X(x) = \sum_k x_k p_X(x_k)$$

$$\exists E[|x|] = \sum_k |x_k p_X(x_k)| < \infty$$

10.4 Variance, Standard Deviation

The **variance** of a random variable X is

$$\sigma_X^2 = \text{VAR}[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

The **Standard Deviation** is

$$\sigma_X = \text{STD}[X] = \sqrt{\text{VAR}[X]}$$

10.5 Expected Value and Variance Properties

$$E[g(X) + h(X)] = E[g(X)] + E[h(X)]$$

$$E[aX] = aE[X] \quad E[X + c] = E[X] + c$$

$$\text{VAR}(cX) = c^2 \text{VAR}(X) \quad \text{VAR}(X + c) = \text{VAR}(X)$$

10.6 Conditional Expected Value

For X a DRV, and suppose we know B has occured,

$$m_{X|B} = E[X|B] = \sum_{x \in S_X} x p_X(x|B)$$

$$= \sum_k x_k P_X(x_k|B)$$

10.7 Conditional Variance

$$\text{VAR}[X|B] = E[(X - m_{X|B})^2|B] =$$

$$\sum_{k=1}^{\infty} (X_k - m_{X|B})^2 p_X(x_k|B) = E[X^2|B] - m_{X|B}^2$$

11 Cumulative Distribution Function

PMF's use events $\{X = b\}$, whereas Cumulative Distribution Functions (CDF) use events $\{X \leq b\}$.

$$F_X(x) = P[X \leq x]$$

11.1 Properties of the CDF

$$0 \leq F_X(x) \leq 1$$

$$\lim_{x \to \infty} F_X(x) = 1 \qquad \lim_{x \to -\infty} F_X(x) = 0$$

$$F_X(a) \leq F_X(b) \; \forall a < b$$

$$F_X(b) = \lim_{h \to 0} F_X(b + h) = F_X(b^+)$$

$$P[a < X \leq b] = F_X(b) - F_X(a)$$

$$P[X = b] = F_X(b) - F_X(b^-)$$

$$P[X > x] = 1 - F_X(x)$$

11.2 CDF of a Discrete RV

$$F_X(x) = \sum_{x_k \leq x} p_X(x_k) = \sum_k P_X(x_k) u(x - x_k)$$

11.3 CDF of a Continuous RV

$$F_X(x) = \int_{-\infty}^{x} f(t) dt$$

11.4 Conditional CDF

$$F_X(x|C) = \frac{P[\{X \leq x\} \cap C]}{P[C]} \quad \text{if } P[C] > 0$$

12 Probability Density Function

$$f_X(x) = \frac{d}{dx} F_X(x)$$

12.1 Properties of the PDF

$$f_X(x) \geq 0 \qquad 1 = \int_{-\infty}^{\infty} f_X(x) dx$$

$$P[a \leq X \leq b] = \int_a^b f_X(x) dx$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t) dt$$

12.2 PDF of a Discrete RV

$$u(x) = \int_{-\infty}^{x} \delta(t) dt$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \sum_k p_X(x_k) \delta(x - x_k)$$

12.3 Conditional PDF

$$f_X(x|C) = \frac{d}{dx} F_X(x|C)$$

12.4 Application of Theorem of Total Probability

Suppose events $B_1, B_2, \ldots, B_n$ partition the sample space S.

$$F_X(x) = \sum_{i=1}^{n} P[X \leq x|B_i]P[B_i]$$

$$= \sum_{i=1}^{n} F_X(x|B_i)P[B_i]$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \sum_{i=1}^{n} f_X(x|B_i)P[B_i]$$

13 Gaussian (Normal) RV

The PDF for the Gaussian Random Variable is given in the table.

13.1 Gaussian CDF

ϕ is the CDF for a standard Gaussian.

$$\phi(z) = \phi\left(\frac{x - m}{\sigma}\right) = P[X \leq x] = F_X(x)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

13.2 Q Function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt$$

$$Q(z) = 1 - \phi(z) = P[X > x]$$

$$Q(0) = 1/2 \quad Q(-x) = 1 - Q(x)$$

13.3 Standard Gaussian RV

To move from any Gaussian to Standard (i.e. $X \sim N(m, \sigma^2) \to z \sim N(0, 1)$), use

$$z = \frac{x - m}{\sigma}$$

14 Other Features of CRV's

14.1 Expected Value

$$E[X] = \int_{-\infty}^{+\infty} t f_X(t) dt$$

14.1.1 Expected Value of Y=g(X)

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

14.1.2 Conditional Expected Value

$$E[X|A] = \int_{-\infty}^{\infty} x f_X(x|A) dx$$

14.2 Variance, Standard Deviation

The **variance** of a random variable X is

$$\text{VAR}[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

The **standard deviation** is

$$\text{STD}[X] = \sqrt{\text{VAR}[X]}$$

14.3 Nth Moment

The **nth moment** of a random variable X is given by

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

15 Functions of RVs - CDF, PDF of Y

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(i)|}$$

$$f_Y(y) = \sum_k \frac{f_X(x)}{dy/dx}\bigg|_{x=x_k} = \sum_k f_X(x) \left|\frac{dx}{dy}\right|\bigg|_{x=x_k}$$

16 Bounds on Probability

16.1 Markov Inequality

Suppose X is a RV with mean $E[X]$. Then

$$P[X \geq a] \leq \frac{E[X]}{a} \quad \text{for X nonnegative}$$

16.2 Chebyshev Inequality

Suppose X is a RV with mean $m = E[X$ and variance σ^2.

$$P[|X - m| \geq a] \leq \frac{\sigma^2}{a^2} \qquad D^2 = (X - m)^2 \longrightarrow$$

$$P[D^2 \geq a^2] \leq \frac{E[(X - m)^2]}{a^2} = \frac{\sigma^2}{a^2}$$

16.3 Chernoff Bound

$$P[X \leq a] = e^{-sa} E[e^{sX}]$$

17 Characteristic Function

$$\phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega) e^{-j\omega x} d\omega$$

17.1 Characteristic Function for DRV's

$$\phi_X(\omega) = \sum_k P_X(x_k) e^{j\omega x_k} \quad , X \text{ a DRV}$$

$$\phi_X(\omega) = \sum_{-\infty}^{\infty} P_X(k) e^{j\omega k} \quad , X \in \mathbb{Z}$$

17.2 Moment Theorem

$$E\left[X^n\right] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \phi_X(\omega)\bigg|_{\omega=0}$$

18 Moment Generating Function

$$M(s) = E[e^{sX}] = \Phi(-js)$$

19 Probability Generating Function

$$G_N(z) = E\left[z^N\right] = \sum_{k=0}^{\infty} p_N(k) z^k$$

19.1 Characteristic Function

$$G_N(e^{j\omega}) = \phi_N(\omega)$$

19.2 PMF Relationship

$$\text{PMF:} \quad p_N(k) = \frac{1}{k!} \frac{d^k}{dz^k} G_N(k)\bigg|_{z=0}$$

20 Laplace Transform of PDF

$$X(s) = \int_0^{\infty} f_X(x) e^{-sx} dx = E[e^{-sX}]$$

$$E[X^n] = (-1)^n \frac{d^n}{ds^n} X(s)\bigg|_{s=0}$$

21 Joint PMF

$$p_{X,Y}(x, y) = P[\{X = x\} \cap \{Y = y\}]$$

$$P[X \text{ in } B] = \sum_{(x_j, y_k) \text{ in } B} \sum p_{X,Y}(x_j, y_k)$$

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k) = 1$$

22 Marginal PMF

$$p_X(x_j) = P[X = x_j] = \sum_{k=1}^{\infty} p_{X,Y}(x_j, y_k)$$

23 Joint CDF

$$F_{X,Y}(x_1, y_1) = P[X \leq x_1, Y \leq Y_1]$$

23.1 Properties of the Joint CDF

$$F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$$

for $x_1 \leq x_2, y_1 \leq y_2$

$$F_{X,Y}(x_1, -\infty) = 0, F_{X,Y}(-\infty, y_1) = 0, F_{X,Y}(\infty, \infty) = 0$$

$$F_X(x_1) = F_{X,Y}(x_1, \infty) \quad F_Y(y_1) = F_{X,Y}(\infty, y_1)$$

$$\lim_{x \to a^+} F_{X,Y}(x, y) = F_{X,Y}(a, y)$$

$$\lim_{x \to b^+} F_{X,Y}(x, y) = F_{X,Y}(x, b)$$

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{X,Y}(x_2, y_2)$$

$$-F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$

24 Joint PDF

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

$$P[X \in B] = \int_B \int f_{X,Y}(x, y) dx dy$$

$$F_{XY}(x, y) = P[X \leq x, Y \leq y]$$

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x, y) dx dy$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$$

25 Marginal PDF

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

25.1 Properties of the Marginal PDF

$$f_X(x) \geq 0 \quad f_Y(y) \geq 0$$

26 Independence of RV's

X and Y are independent if for any $X \in A$, $Y \in B$

$$P[X \in A, Y \in B] = P[X \in A]P[Y \in B]$$

If X, Y independent, then

$$p_{XY}(x_j, y_k) = P[X = x_j, Y = y_k] =$$

$$P[X = x_j]P[Y = y_k] = p_X(x_j) p_Y(y_j)$$

X, Y independent iff

$$F_{XY}(x, y) = F_X(x) F_Y(y)$$

$$f_{XY}(x, y) = f_X(x) f_Y(y) \text{ if } X, Y \text{ jointly cont.}$$

27 Expected Value for Functions of 2 RVs

If X, Y discrete:

$$E[X] = g(x_j, y_k) p_{XY}(x_j, y_k)$$

If X, Y continuous:

$$E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) dx dy$$

$$E[X+Y]=E[X]+E[Y]$$

27.1 Expected Value and Independence

Let $g(X,Y)=g_1(X)g_2(Y)$, and X,Y independent

$$Z=XY \leftrightarrow E[Z]=E[XY]=E[X]E[Y]$$

$$E[g(X,Y)]=E[g_1(X)]E[g_2(Y)]$$

28 Joint Moment

If X,Y discrete:

$$E[X^jY^k]=\sum_i\sum_n x_i^j y_n^k p_{XY}(x_i,y_n)$$

If X,Y jointly continuous:

$$E[X]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{XY}(x,y)dxdy$$

28.1 Correlation

$$E[XY]=E[X^{j=1}Y^{k=1}]$$

If $E[XY]=0$, then X,Y are orthogonal.

28.2 Central Moment

$$E[(X-E[X])^j\cdot(Y-E[Y])^k]$$

28.2.1 Variance

$$\text{VAR}(X)=E[(X-E[X])^2\cdot(Y-E[Y])^0]$$

$$\text{VAR}(X)=E[(X-E[X])^2]$$

29 Covariance

$$\text{COV}(X,Y)=E[(X-E[X])(Y-E[Y])]$$

$$=E[(X-E[X])^1\cdot(Y-E[Y])^1]=E[XY]-E[X]E[Y]$$

If $E[X]=0$ and/or $E[Y]=0$, then

$$\text{COV}(X,Y)=E[XY]$$

29.1 Correlation Coefficient

$$\rho_{XY}=\frac{\text{COV}(X,Y)}{\sigma_X\sigma_Y},\quad -1\le\rho_{XY}\le 1$$

If X,Y uncorrelated, then

$$\text{COV}(X,Y)=0,\quad E[XY]=E[X]E[Y],\quad \rho_{XY}=0$$

If X,Y **independent**, then they are uncorrelated.

29.2 Covariance Properties

$$\text{COV}(X,X)=\text{VAR}(X)\quad \text{COV}(X,Y)=\text{COV}(Y,X)$$

$$\text{COV}(\alpha X,Y)=\alpha\text{COV}(X,Y)$$

$$\text{COV}(X+c,Y)=\text{COV}(X,Y)$$

$$\text{COV}(X+Y,Z)=\text{COV}(X,Z)+\text{COV}(Y,Z)$$

30 Conditional Probabilities

30.1 Case 1: X, Y Discrete - Conditional PMF

$$p_Y(y|x)=P[Y=y|X=x]=$$

$$=\frac{P[X=x,Y=y]}{P[X=x]}=\frac{p_{XY}(x,y)}{p_X(x)}$$

$$p_Y(y_k|x_j)=\frac{p_{XY}(x_j,y_k)}{p_X(x_j)}\longrightarrow$$

$$p_{XY}(x_j,y_k)=p_Y(y_k|x_j)\cdot p_X(x_j)$$

$$P[Y\in A|X=x_k]=\sum_{y_j\in A}p_Y(y_j|x_k)$$

$$P[Y\in A]=\sum_{x_k}P[Y\in A|X=x_k]p_X(x_k)$$

30.2 Case 2: X discrete, Y continuous - Conditional PDF

$$F_Y(y|x_k)=\frac{P[Y\le y,X=x_k]}{P[X=x_k]},\quad P[X=x_k]>0$$

$$f_Y(y|x_k)=\frac{d}{dy}F_Y(y|x_k)$$

If X,Y independent,

$$P[Y\in A|X=x_k]=\int_{y\in A}f_Y(y|x_k)dy$$

30.3 Case 3: X, Y continuous - Conditional PDF

$$f_Y(y|x)=\frac{d}{dy}F_Y(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}$$

$$P[Y\in A|X=x]=\int_{y\in A}f_Y(y|x)dy$$

$$P[Y\in A]=\int_{-\infty}^{\infty}P[Y\in A|X=x]f_X(x)dx$$

If X,Y independent,

$$f_Y(y|x)=\frac{f_X(x)f_Y(y)}{f_X(x)}=f_Y(y)$$

30.4 Bayes Rule

$$f_Y(y|x)=\frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_{XY}(x,y)=f_Y(y|x)f_X(x)=f_X(x|y)f_Y(y)$$

$$f_Y(y|x)=\frac{f_X(x|y)f_Y(y)}{f_X(x)}$$

31 Conditional Expectation

31.1 X,Y Discrete

$$E[Y|x]=\sum_{y_k}p_Y(y_k|x)$$

31.2 X,Y Continuous

$$E[Y|x]=\int_{-\infty}^{\infty}yf_Y(y|x)dy$$

31.3 Law of total Expectation

Since $E[Y|x]=g(X)$, we define $E[g(x)]$

$$E[E[Y|x]]=\int_{-\infty}^{\infty}E[Y|x]f_X(x)dx=E[Y]$$

for any function $h(Y)$, where $E[h(Y)]=E[E[h(Y|x)]]$

$$E[Y^k]=E[E[Y^k|x]]$$

32 Functions of Two RVs

Let $Z=g(X,Y)$ (function of two RVs). Then,

$$F_Z(z)=P[X\in R_z]=\iint_{(x,y)\in R_z}f_{XY}(x,y)dxdy$$

$$f_Z(z)=\frac{d}{dz}F_Z(z)=\int_{-\infty}^{\infty}f_{XY}(x,z-x)dx$$

If X,Y independent, then

$$f_Z(z)=f_X(x)*f_Y(y)=\int_{-\infty}^{\infty}f_X(x)f_Y(z-x)dx$$

33 Transformations of Two RVs

Let $W=(X,Y)$ and $Z_1=g_1(W)$ and $Z_2=g_2(W)$

$$F_{Z_1,Z_2}(z_1,z_2)=P[g_1(W)\le z_1,g_2(X)\le z_2]$$

$$F_{Z_1,Z_2}(z_1,z_2)=\iint_{W:g_k(W)\le z_k}f_{XY}(x,y)dxdy$$

34 Linear Transformations

$$\begin{bmatrix}V\\W\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}=A\begin{bmatrix}X\\Y\end{bmatrix}$$

Assume A is invertible:

$$\begin{bmatrix}X\\Y\end{bmatrix}=A^{-1}\begin{bmatrix}V\\W\end{bmatrix}$$

34.1 Joint PDF of Linear Transformation

Let $Z=g(X,Y)$. The vector Z is:

$$Z=AW\quad Z=\begin{bmatrix}V\\W\end{bmatrix}\quad W=\begin{bmatrix}X\\Y\end{bmatrix}$$

The Joint PDF of Z is

$$f_Z(z)=\frac{f_W(A^{-1}z)}{|A|}\quad |A|=\det\begin{bmatrix}a&b\\c&d\end{bmatrix}$$

35 Joint Gaussian RVs

The random variables X,Y are jointly gaussian if:

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{XY}^2}}\exp(A)$$

$$A=\frac{-1}{2(1-\rho_{XY}^2)}\left[\left(\frac{x-m_1}{\sigma_1}\right)^2-\right.$$

$$\left.2\rho_{XY}\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-m_2}{\sigma_2}\right)+\left(\frac{y-m_2}{\sigma_2}\right)^2\right]$$

35.1 Joint Standard (Normal) Gaussians

If $X\ N(0,1)$, $Y\ N(0,1)$, then

$$f_{XY}(x,y)=\frac{1}{2\pi\sqrt{1-\rho_{XY}^2}}\exp(A)$$

$$A=\frac{1}{2(1-\rho^2)}\left(x^2-2\rho_{XY}\cdot xy+y^2\right)$$

$$f_{XY}(x,y)=g(r)=C\exp\left[\frac{-r^2}{2\sigma^2}\right]$$

35.2 Independence (m=0,σ=1)

If X,Y independent $\leftrightarrow$

$$\text{COV}(X,Y)=0\quad \rho_{XY}(x,y)=0$$

$$f_{XY}(x,y)=\frac{1}{2\pi}\exp\left(-\frac{1}{2}(x^2+y^2)\right)$$

35.3 Independence (m=0)

If $X\ N(0,1)$, $Y\ N(0,1)$, then

$$f_{XY}(x,y)=\frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}(x^2+y^2)\right)$$

35.4 Constant A

If A (exponent of Joint Gaussian) is a constant K

$$K=\left[\left(\frac{x-m_1}{\sigma_1}\right)^2-\ldots+\left(\frac{y-m_2}{\sigma_2}\right)^2\right]$$

$$f_XY(x,y)=C\exp\left[-\frac{1}{2(1-\rho^2)}K\right]=\text{constant}$$

35.5 Major Axis

If X,Y not independent, then the principal axes has

$$\theta=\frac{1}{2}\arctan^{-1}\tan\left(\frac{2\rho_{XY}\sigma_1\sigma_2}{\sigma_1^2-\sigma_2^2}\right)$$

35.6 Conditional PDF

The conditional PDF of X given $Y=y$ is

$$f_X(x|y)=\frac{f_{XY}(x,y)}{f_Y(y)}=\frac{1}{2\pi\sigma_1^2\sqrt{1-\rho_{XY}^2}}\cdot$$

$$\exp\left(\frac{-1}{2(1-\rho_{XY}^2\sigma_1^2)}\left[x-\rho_{XY}\frac{\sigma_1}{\sigma_2}(y-m_2)-m_1\right]^2\right)$$

35.7 Conditional Expectation

$$E[(X-m_1)(Y-m_2)|Y]=(y-m_2)E[X-m_1|Y=y]$$

$$=(y-m_2)\left(\rho_{XY}\frac{\sigma_1}{\sigma_2}(y-m_2)\right)=\rho_{XY}\frac{\sigma_1}{\sigma_2}(y-m_2)^2$$

35.8 Covariance

$$\text{COV}(X,Y)=E[(X-m_1)(Y-m_2)]$$

$$=E[E[(X-m_1)(Y-m_2)|Y]]=\rho_{XY}\sigma_1\sigma_2$$

36 Sum of RVs

Let $X_1,X_2,...,X_n$ be a sequence of RVs, with

$$S_n=X_1+X_2+\ldots+X_n$$

36.1 Mean and Variance of Sum of RVs

$$E[X_1+X_2+\ldots+X_n]=E[X_1]+E[X_2]+\ldots+E[X_n]$$

$$\text{VAR}(X_1+\ldots+X_n)=\sum_{k=1}^{n}\text{VAR}(X_k)$$

$$+\sum_{j=1}\sum_{k=1}\text{COV}(X_j,X_k),\quad j\ne k$$

If $X_1,X_2,...,X_n$ independent, then

$$\text{VAR}(X_1+\ldots+X_n)=\text{VAR}(X_1)+\ldots+\text{VAR}(X_n)$$

36.2 PDF of Sums of Independent RVs

Let $X_1,X_2,...,X_n$ independent, then

$$\phi_{S_n}(\omega)=E[e^{j\omega S_n}]=E[e^{j\omega(X_1+X_2+...+X_n)}]$$

$$E[e^{j\omega X_1}]\ldots E[e^{j\omega X_n}]=\phi_{X_1}(\omega)\ldots\phi_{X_n}(\omega)$$

$$f_{S_n}=\mathcal{F}^{-1}\left[\phi_{X_1}(\omega)\ldots\phi_{X_n}(\omega)\right]$$

37 Independent Identically Distributed RVs (iid)

If $X_1,X_2,...,X_n$ iid RVs, with

$$E[X_j]=m_x\quad \text{VAR}(X_j)=\sigma_x^2\quad \text{for } j=1,...,n$$

37.1 Mean and Variance of iid RVs

$$E[S_n]=E[X_1]+\ldots+E[X_n]=n\cdot m_x$$

$$\text{VAR}(S_n)=n\cdot\text{VAR}(X_j)=n\sigma_x^2$$

37.2 PDF of iid RVs

$$\phi_{X_k}(\omega)=\phi_X(\omega),k=1,...,n\leftrightarrow\phi_{S_N}(\omega)=[\phi_X(\omega)]^n$$

$$f_{S_n}=\mathcal{F}^{-1}(\phi_{S_n}(\omega))=\mathcal{F}^{-1}(\phi_X(\omega)^n)$$

38 Sample Mean

$$M_n=\frac{1}{n}\sum_{j=1}^{n}X_j$$

38.1 Expected Value and Varianceof Sample Mean

$$E[M_n]=E\left[\frac{1}{n}\sum_{j=1}^{n}X_j\right]\to E[M_n]=\frac{1}{n}\sum_{j=1}^{n}E[X_j]$$

$$\text{VAR}(M_n)=E\left[(M_n-E[M_n])^2\right]=\text{VAR}(S_n)/n^2$$

if $X_1,\ldots,X_n$ iid RVs:

$$E[M_n]=m_x\leftrightarrow E[S_n]=n\cdot m_x$$

$$\text{VAR}(S_n)=n\sigma^2\leftrightarrow\text{VAR}(M_n)=\frac{\sigma^2}{n}$$

38.2 Sample Mean Chebyshev Bound

$$P[|Z-E[Z]|\ge\epsilon]\le\frac{\text{VAR}(Z)}{\epsilon^2}\quad,\epsilon>0$$

$$P[|M_n-m_x|\ge\epsilon]\le\frac{\sigma^2}{n\epsilon^2}$$

$$P[|M_n-m_x|<\epsilon]\ge 1-\frac{\sigma^2}{n\epsilon^2}$$

39 Laws of Large Numbers

$$\text{Weak Law}:\quad\lim_{n\to\infty}P[|M_n-m_x|<\epsilon]=1$$

$$\text{Strong Law}:\quad P\left[\lim_{n\to\infty}M_n=m_x\right]=1$$

40 Central Limit Theorem

Let $S_n=X_1,X_2,...,X_n$ iid RVs

$$\lim_{n\to\infty}P[Z_n\le z]=\frac{1}{\sqrt{2\pi}}\int_{\infty}^{z}e^{-x^2/2}dx$$

41 Common Sums

$$\sum_{k=0}^{n}a^k=\frac{1-a^{n+1}}{1-a},a\ne1$$

$$\sum_{k=0}^{n}ka^k=\frac{a}{(1-a)^2}[1-(n+1)a^n+na^{n+1}],a\ne1$$

$$\sum_{k=0}^{n}k=\frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n}k^2=\frac{(n(n+1)(2n+1)}{6}$$

$$\sum_{k=0}^{n}k^3=\frac{(n^2(n+1)^2}{4}$$

$$\sum_{k=0}^{\infty}a^k=\frac{1}{1-a}$$

$$\sum_{k=0}^{\infty}ka^k=\frac{a}{(1-a)^2}$$

$$\sum_{k=0}^{\infty}k^2a^k=\frac{a^2+a}{(1-a)^3}$$

42 Common Discrete Random Variables

RV	S	$p_X(k)$	$E[X]$	$\text{VAR}[X]$	$G_X(z)$
Uniform	$\{1,2,...,L\}$	$\frac{1}{L}$	$\frac{L+1}{2}$	$\frac{L^2-1}{2}$	$\frac{z}{L}\frac{1-z^L}{1-z}$
Bernoulli	$\{0,1\}$	$(1-p)p$	p	$p(1-p)$	$q+pz$
Binomial	$\{0,1,...,n\}$	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$	$(q+pz)^n$
-ve Binomial	$\{r,r+1,...\}\ r\in\mathbb{R}$	$\binom{k-1}{r-1}p^r(1-p)^{k-r}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{pz}{1-qz}\right)^r$
Geometric v1	$\{0,1,2,...\}$	$(1-p)^kp$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-qz}$
Geometric v2	$\{1,2,...\}$	$(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pz}{1-qz}$
Poisson	$\{0,1,2,...\}$	$\frac{\alpha^k}{k!}e^{-\alpha}\quad k=0,1,...,\alpha>0$	α	α	$e^{\alpha(z-1)}$
Zipf	$\{1,2,...,L\}$	$\frac{1}{c_L}\frac{1}{k},\ c_L=\sum_{j=1}^{L}\frac{1}{j}$	$\frac{L}{c_L}$	$\frac{L(L+1)}{2c_L}-\frac{L^2}{c_L^2}$	

43 Common Continuous Random Variables

RV	S	$f_X(x)$	$E[X]$	$\text{VAR}[X]$	$\phi_X(\omega)$	$F_X(x)$
Uniform	$[a,b]$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{j\omega b}-e^{j\omega a}}{j\omega(b-a)}$	$\frac{x-a}{b-a},x\in[a,b]$
Exponential	$[0,\infty)$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-j\omega}$	$1-e^{-\lambda x}$
Gaussian (Normal)	$(-\infty,\infty)$	$\frac{e^{-(x-m)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$	m	σ^2	$e^{jm\omega-\sigma^2\omega^2/2}$	$\Phi(x)$
Gamma	$(0,\infty)$	$\frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{1}{1-j\omega/\lambda}\right)^\alpha\ \alpha>0,\lambda>0$	
$m-1$ Erlang	$(0,\infty)$	$\frac{\lambda e^{-\lambda x}(\lambda x)^{m-2}}{(m-1)!}$			$\left(\frac{1}{1-j\omega/\lambda}\right)^m$	
χ-Squared (k DoF)	$(0,\infty)$	$\frac{x^{(k-2)/2}e^{-x/2}}{2^{k/2}\Gamma(k/2)}$			$\left(\frac{1}{1-2j\omega}\right)^{k/2}$	
Laplacian	$(-\infty,\infty)$	$\frac{\alpha}{2}e^{-\alpha\lvert x\rvert}$	0	$\frac{2}{\alpha^2}$	$\frac{\alpha^2}{\omega^2+\alpha^2}$	$\begin{cases}\frac{1}{2}e^{-x\alpha} & x\le0\\ 1-\frac{1}{2}e^{-x\alpha} & x\ge0\end{cases}$
Rayleigh	$[0,\infty)$	$\frac{x}{\alpha^2}e^{-x^2/2\alpha^2}$	$\alpha\sqrt{\pi/2}$	$(2-\pi/2)\alpha^2$		$1-e^{-x^2/(2\alpha^2)}$
Pareto	$[x_m,\infty)$	$\alpha\frac{x_m^\alpha}{x^{\alpha+1}},x\ge x_m$, 0 else	$\frac{\alpha x_m}{\alpha-1}$	$\frac{\alpha x_m^2}{(\alpha-2)(\alpha-1)^2}$		$1-\left(\frac{x_m}{x}\right)^\alpha$
Cauchy	$(-\infty,+\infty)$	$\frac{\alpha/\pi}{x^2+\alpha^2}$			$e^{-\alpha\lvert w\rvert}$	
Beta	$0<x<1$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$		

44 Q(X) for STANDARD Gaussian

x	$Q(x)$	Approx.	x	$Q(x)$	Approx.
0	5.00E-01	5.00E-01	2.7	3.47E-03	3.46E-03
0.1	4.60E-01	4.58E-01	2.8	2.56E-03	2.55E-03
0.2	4.21E-01	4.17E-01	2.9	1.87E-03	1.86E-03
0.3	3.82E-01	3.78E-01	3.0	1.35E-03	1.35E-03
0.4	3.45E-01	3.41E-01	3.1	9.68E-04	9.66E-04
0.5	3.09E-01	3.05E-01	3.2	6.87E-04	6.86E-04
0.6	2.74E-01	2.71E-01	3.3	4.83E-04	4.83E-04
0.7	2.42E-01	2.39E-01	3.4	3.37E-04	3.36E-04
0.8	2.12E-01	2.09E-01	3.5	2.33E-04	2.32E-04
0.9	1.84E-01	1.82E-01	3.6	1.59E-04	1.59E-04
1.0	1.59E-01	1.57E-01	3.7	1.08E-04	1.08E-04
1.1	1.36E-01	1.34E-01	3.8	7.24E-05	7.23E-05
1.2	1.15E-01	1.14E-01	3.9	4.81E-05	4.81E-05
1.3	9.68E-02	9.60E-02	4.0	3.17E-05	3.16E-05
1.4	8.08E-02	8.01E-02	4.5	3.40E-06	3.40E-06
1.5	6.68E-02	6.63E-02	5.0	2.87E-07	2.87E-07
1.6	5.48E-02	5.44E-02	5.5	1.90E-08	1.90E-08
1.7	4.46E-02	4.43E-02	6.0	9.87E-10	9.86E-10
1.8	3.59E-02	3.57E-02	6.5	4.02E-11	4.02E-11
1.9	2.87E-02	2.86E-02	7.0	1.28E-12	1.28E-12
2.0	2.28E-02	2.26E-02	7.5	3.19E-14	3.19E-14
2.1	1.79E-02	1.78E-02	8.0	6.22E-16	6.22E-16
2.2	1.39E-02	1.39E-02	8.5	9.48E-18	9.48E-18
2.3	1.07E-02	1.07E-02	9.0	1.13E-19	1.13E-19
2.4	8.20E-03	8.17E-03	9.5	1.05E-21	1.05E-21
2.5	6.21E-03	6.19E-03	10.0	7.62E-24	7.62E-24
2.6	4.66E-03	4.65E-03			