ECE302 Course Notes

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September 2021

1 Set Theory Review

1.1 Union and Intersection

$$A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$$

$$A \cap B = \{x : x \in A \quad \text{and} \quad x \in B\}$$

1.2 Complement

$$A^C = \{x : x \notin A\}$$

1.3 Disjoint Sets

Two sets A_i and A_j are disjoint if

$$A_i \cap A_j = \varnothing \quad \forall i, j \; i \neq j$$

1.4 Collectively Exhaustive Sets

Sets $A_i, ..., A_n$ are collectively exhaustive if

$$\cup_{i=1}^{N} A_i = S$$

1.5 Partition

Sets $A_i, ..., A_n$ are called a partition of S if $A_i, ..., A_n$ are disjoint and collectively exhaustive.

1.6 Properties of Sets

1.6.1 Commutative

$$A \cap B = B \cap A \qquad A \cup B = B \cup A$$

1.6.2 Associative

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

1.6.3 Distributive

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

1.7 Relative Complement/Difference

$$A - B = \{x : x \in A \quad \text{and} \quad x \notin B\}$$
$$A - B = A \cap B^C$$

1.7.1 DeMorgan's

$$(A \cup B)^C = A^C \cap B^C \quad (A \cap B)^C = A^C \cup B^C$$

2 Probability Theory Introduction

2.1 Relative Frequency

Suppose that an experiment is repeated n times under identical conditions. Let $N_0(n), N_1(n), ..., N_k(n)$ be the number of times the outcome k happens. Then the relative frequency of outcome k is

$$f_k(n) = \frac{N_k(n)}{n} \text{ where } \lim_{n \to \infty} f_k(n) = p_k$$

2.2 Axioms of Probability

$$P(A) \geq 0 \qquad\qquad P[S] = 1$$
$$A \cap B = \varnothing \quad \longrightarrow \quad P(A \cup B) = P(A) + P(B)$$
$$P\left[\bigcup_{k=1}^{\infty} A_k\right] = \sum_{k=1}^{\infty} P[A_k]$$

If A_1, A_2 is a sequence of events s.t. $A_i \cap A_j = \varnothing \quad i \neq j$

2.3 Bayesian Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

3 Counting Methods and Sampling

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Permutations of n distinct objects (k-tuples):	$n!$
Number of **ordered** samples with size k **with** replacement:	n^k
Number of **ordered** samples with size k **without** replacement:	$\frac{n!}{(n-k)!}$
Number of **unordered** samples with size k **without** replacement:	$nk = \frac{n!}{k!(n-k)!}$
Number of **unordered** samples with size k and **with** replacement:	$\binom{n-1+k}{k} = \binom{n-1+k}{n-1}$

3.1 Binomial Coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \binom{n}{k} = \binom{n}{n-k}$$

4 Conditional Probability

If A and B are related, then the conditional probability of A *given* that B (and $P[B] > 0$) has occurred is

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

5 Theorem of Total Probability

For $B_1, B_2, ..., B_n$ mutually exclusive events whose union equals the sample space S (e.g. $B_1, ..., B_n$ is a **partition** of S), then

$$P[A] = P[A|B_1]P[B_1]... + P[A|B_n]P[B_n]$$

6 Bayes Rule

For $B_1, B_2, ..., B_n$ a partition of sample space S,

$$P[B_j|A] = \frac{P[A \cap B_j]}{P[A]} = \frac{P[A|B_j]P[B_j]}{\sum_{k=1}^{n} P[A|B_k]P[B_k]}$$

7 Independence

If knowledge of the occurrence of event B does not alter the probability of event A, then event A is independent of B.

$$P[A] = P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Define A, B to be independent if

$$P[A \cap B] = P[A]P[B]$$

Then

$$P[A|B] = P[A]\ ,\ P[B|A] = P[B]$$

$$P[A^C \cap B^C] = P[A^C]P[B^C]$$

7.1 Notes on Independence

If two events have nonzero probability ($P[A] > 0, P[B] > 0$), and are mutually exclusive, then they cannot be independent

7.2 Triplet Independence

For three events A, B, C to be independent,

- A,B,C Pairwise Independent
- knowledge of occurrence of any two events (e.g. A, B) should not effect the prob of the third (C)

7.2.1 Pairwise Independence

$$P[A \cap B] = P[A]P[B] \quad P[A \cap C] = P[A]P[C]$$

$$P[B \cap C] = P[B]P[C]$$

7.2.2 Independence of Events

$$P[C|A \cap B] = \frac{P[A \cap B \cap C]}{P[A \cap B]} = P[C]$$

Finally, for **Triplet Independence**, we must have

$$P[A \cap B \cap C] = P[A]P[B]P[C]$$

8 Sequential Experiments

8.1 Bernoulli Trials

Let k be the num of successes in n independent Bernoulli trials. Then the probabilities of k are given by **binomial probability law**

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for} \quad k = 0, ..., n$$

8.2 Multinomial Probability Law

Let $B_1, B_2, ..., B_M$ be a partition of the sample space S, and let $P[B_j] = p_j$. Also, the events are disjoint:

$$p_1 + p_2 + ... + p_M = 1$$

The multinomial probability law is

$$P[(k_1, ..., k_M)] = \frac{n!}{k_1!...k_M!} p_1^{k_1} ... p_M^{k_M}$$

8.3 Geometric Probability Law

The probability that more than K trials are required before a success (with probability $p, q = 1 - p$) occurs in a series of repeated independent Bernoulli trials is

$$P[m > K] = p \sum_{m=K+1}^{\infty} q^{m-1} = pq^K \frac{1}{1-q} = q^K$$

The probability that K trials are required for a success (with probability $p, q = 1 - p$) is

$$P[m = K] = (p)(1-p)^{(K-1)} = pq^{(K-1)}$$

8.3.1 Hypergeometric Distribution

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Where K is the number of success in the population, k is the number of observed successes, N is the population size, and n is the sample size.

9 Random Variables (RV)

A **Random Variable** X is a function that assigns a real number $X(\zeta)$ to each outcome ζ in the sample space of a random experiment.

10 Discrete Random Variable (DRV)

A **Discrete Random Variable X** is defined as a random variable that assumes values from a countable set.

10.1 PMF

$$p_X(x) = P[X = x] = P[\{\zeta : X(\zeta) = x\} \quad x \in \mathbb{R}$$

For x_k in S_X, $\quad p_X(x_k) = P[A_k]$

10.1.1 PMF Properties

$$p_X(x) \geq 0 \quad \forall x$$

$$\sum_{x \in S_X} p_X(x) = \sum_k p_X(x_k) = \sum_k P[A_k] = 1$$

$$P[X \text{in} B] = \sum_{x \in B} p_X(x) \quad \text{where } B \subset S_X$$

10.2 Conditional PMF

Let X be a DRV with PMF $P_X(x)$, and $\exists C, P[C] > 0$. The **Conditional PMF** is given by

$$p_X(x|C) = P[X = x|C] = \frac{P[\{X = x\} \cap C]}{P[C]}$$

10.3 Expected Value

The **expected value** (or **mean**) of a DRV is

$$E[X] = \sum_{x \in S_X} x p_X(x) = \sum_k x_k p_X(x_k)$$

$$\exists E[|x|] = \sum_k |x_k p_X(x_k)| < \infty$$

10.4 Variance, Standard Deviation

The **variance** of a random variable X is

$$\sigma_X^2 = \text{VAR}[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

The **Standard Deviation** is

$$\sigma_X = \text{STD}[X] = \sqrt{\text{VAR}[X]}$$

10.5 Expected Value and Variance Properties

$$E[g(X) + h(X)] = E[g(X)] + E[h(X)]$$
$$E[aX] = aE[X] \quad E[X + c] = E[X] + c$$

$$(cX) = c^2(X) \quad (X + c) = (X)$$

10.6 Conditional Expected Value

For X a DRV, and suppose we know B has occured,

$$m_{X|B} = E[X|B] = \sum_{x \in S_x} x p_X(x|B)$$
$$= \sum_k x_k P_X(x_k|B)$$

10.7 Conditional Variance

$$\text{VAR}[X|B] = E[(X - m_{X|B})^2|B] =$$
$$\sum_{k=1}^{\infty}(X_k - m_{X|B})^2 p_X(x_k|B) = E[X^2|B] - m_{X|B}^2$$

11 Cumulative Distribution Function

PMF's use events $\{X = b\}$, whereas Cumulative Distribution Functions (CDF) use events $\{X \leq b\}$.

$$F_X(x) = P[X \leq x]$$

11.1 Properties of the CDF

$$0 \leq F_X(x) \leq 1$$
$$\lim_{x \to \infty} F_X(x) = 1 \qquad \lim_{x \to -\infty} F_X(x) = 0$$
$$F_X(a) \leq F_X(b) \;\; \forall a < b$$
$$F_X(b) = \lim_{h \to 0} F_X(b + h) = F_X(b^+)$$

$$P[a < X \leq b] = F_X(b) - F_X(a)$$
$$P[X = b] = F_X(b) - F_X(b^-)$$
$$P[X > x] = 1 - F_X(x)$$

11.2 CDF of a Discrete RV

$$F_X(x) = \sum_{x_k \leq x} p_X(x_k) = \sum_k P_X(x_k)u(x - x_k)$$

11.3 CDF of a Continuous RV

$$F_X(x) = \int_{-\infty}^{x} f(t)dt$$

11.4 Conditional CDF

$$F_X(x|C) = \frac{P[\{X \leq x\} \cap C]}{P[C]} \text{ if } P[C] > 0$$

12 Probability Density Function

$$f_X(x) = \frac{d}{dx} F_X(x)$$

12.1 Properties of the PDF

$$f_X(x) \geq 0 \qquad 1 = \int_{-\infty}^{\infty} f_X(x)dx$$

$$P[a \leq X \leq b] = \int_a^b f_X(x)dx$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

12.2 PDF of a Discrete RV

$$u(x) = \int_{-\infty}^{x} \delta(t)dt$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \sum_k p_X(x_k)\delta(x - x_k)$$

12.3 Conditional PDF

$$f_X(x|C) = \frac{d}{dx} F_X(x|C)$$

12.4 Application of Theorem of Total Probability

Suppose events $B_1, B_2, ..., B_n$ partition the sample space S.

$$F_x(x) = \sum_{i=1}^{n} P[X \leq x | B_i] P[B_i]$$

$$= \sum_{i=1}^{n} F_X(x|B_i) P[B_i]$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \sum_{i=1}^{n} f_X(x|B_i) P[B_i]$$

13 Gaussian (Normal) RV

The PDF for the Gaussian Random Variable is given in the table.

13.1 Gaussian CDF

ϕ is the CDF for a standard Gaussian.

$$\phi(z) = \phi\left(\frac{x-m}{\sigma}\right) = P[X \leq x] = F_X(x)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

13.2 Q Function

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt$$

$$Q(z) = 1 - \phi(z) = P[X > x]$$

$$Q(0) = 1/2 \quad Q(-x) = 1 - Q(x)$$

13.3 Standard Gaussian RV

To move from any Gaussian to Standard (i.e. $X \sim N(m, \sigma^2) \rightarrow z \sim N(0, 1)$), use

$$z = \frac{x-m}{\sigma}$$

14 Other Features of CRV's

14.1 Expected Value

$$E[X] = \int_{-\infty}^{+\infty} t f_X(t) dt$$

14.1.1 Expected Value of Y=g(X)

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

14.1.2 Conditional Expected Value

$$E[X|A] = \int_{-\infty}^{\infty} x f_X(x|A) dx$$

14.2 Variance, Standard Deviation

The **variance** of a random variable X is

$$\text{VAR}[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

The **standard deviation** is

$$\text{STD}[X] = \sqrt{\text{VAR}[X]}$$

14.3 Nth Moment

The **nth moment** of a random variable X is given by

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

15 Functions of RVs - CDF, PDF of Y

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(i)|}$$

$$f_Y(y) = \sum_k \left. \frac{f_X(x)}{dy/dx} \right|_{x=x_k} = \sum_k f_X(x) \left| \frac{dx}{dy} \right| \Bigg|_{x=x_k}$$

16 Bounds on Probability

16.1 Markov Inequality

Suppose X is a RV with mean $E[X]$. Then

$$P[X \geq a] \leq \frac{E[X]}{a} \text{ for X nonnegative}$$

16.2 Chebyshev Inequality

Suppose X is a RV with mean $m = E[X$ and variance σ^2.

$$P[|X - m| \geq a] \leq \frac{\sigma^2}{a^2} \qquad D^2 = (X - m)^2 \longrightarrow$$

$$P[D^2 \geq a^2] \leq \frac{E[(X - m)^2]}{a^2} = \frac{\sigma^2}{a^2}$$

16.3 Chernoff Bound

$$P[X \leq a] = e^{-sa} E[e^{sX}]$$

17 Characteristic Function

$$\phi_X(\omega) = E\left[e^{j\omega X}\right] = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega) e^{-j\omega x} d\omega$$

17.1 Characteristic Function for DRV's

$$\phi_X(\omega) = \sum_k P_X(x_k) e^{j\omega x_k} \quad , X \text{ a DRV}$$

$$\phi_X(\omega) = \sum_{-\infty}^{\infty} P_X(k) e^{j\omega k} \quad , X \in \mathbb{Z}$$

17.2 Moment Theorem

$$E\left[X^n\right] = \frac{1}{j^n} \frac{d^n}{d\omega^n} \phi_X(\omega) \bigg|_{\omega=0}$$

18 Moment Generating Function

$$M(s) = E[e^{sX}] = \Phi(-js)$$

19 Probability Generating Function

$$G_N(z) = E\left[z^N\right] = \sum_{k=0}^{\infty} p_N(k) z^k$$

19.1 Characteristic Function

$$G_N(e^{j\omega}) = \phi_N(\omega)$$

19.2 PMF Relationship

$$\text{PMF:} \quad p_N(k) = \frac{1}{k!}\frac{d^k}{dz^k}G_N(k)\Big|_{z=0}$$

20 Laplace Transform of PDF

$$X(s) = \int_0^\infty f_X(x)e^{-sx}dx = E[e^{-sX}]$$

$$E[X^n] = (-1)^n\frac{d^n}{ds^n}X(s)\Big|_{s=0}$$

21 Joint PMF

$$p_{X,Y}(x,y) = P[\{X=x\}\cap\{Y=y\}]$$

$$P[X \text{ in } B] = \sum_{(x_j,y_k) \text{ in } B}\sum p_{X,Y}(x_j,y_k)$$

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} p_{X,Y}(x_j,y_k) = 1$$

22 Marginal PMF

$$p_X(x_j) = P[X = x_j] = \sum_{k=1}^{\infty} p_{X,Y}(x_j,y_k)$$

23 Joint CDF

$$F_{X,Y}(x_1,y_1) = P[X \le x_1, Y \le Y_1]$$

23.1 Properties of the Joint CDF

$$F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2)$$

for $x_1 \leq x_2, y_1 \leq y_2$

$$F_{X,Y}(x_1, -\infty) = 0, F_{X,Y}(-\infty, y_1) = 0, F_{X,Y}(\infty, \infty) = 0$$

$$F_X(x_1) = F_{X,Y}(x_1, \infty) \quad F_Y(y_1) = F_{X,Y}(\infty, y_1)$$

$$\lim_{x \to a^+} F_{X,Y}(x, y) = F_{X,Y}(a, y)$$

$$\lim_{x \to b^+} F_{X,Y}(x, y) = F_{X,Y}(x, b)$$

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{X,Y}(x_2, y_2)$$

$$-F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$

24 Joint PDF

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

$$P[X \in B] = \int_B \int f_{X,Y}(x, y) dx dy$$

$$F_{XY}(x, y) = P[X \leq x, Y \leq y]$$

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x, y) dx dy$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$$

25 Marginal PDF

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

25.1 Properties of the Marginal PDF

$$f_X(x) \geq 0 \quad f_Y(y) \geq 0$$

26 Independence of RV's

X and Y are independent if for any $X \in A$, $Y \in B$

$$P[X \in A, Y \in B] = P[X \in A]P[Y \in B]$$

If X, Y independent, then

$$p_{XY}(x_j, y_k) = P[X = x_j, Y = y_k] =$$
$$P[X = x_j]P[Y = y_k] = p_X(x_j)p_Y(y_j)$$

X, Y independent iff

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$
$$f_{XY}(x, y) = f_X(x)f_Y(y) \text{ if } X, Y \text{ jointly cont.}$$

27 Expected Value for Functions of 2 RVs

If X, Y discrete:

$$E[X] = g(x_j, y_k)p_{XY}(x_j, y_k)$$

If X, Y continuous:

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f_{XY}(x, y)dxdy$$

$$E[X + Y] = E[X] + E[Y]$$

27.1 Expected Value and Independence

Let $g(X, Y) = g_1(X)g_2(Y)$, and X, Y independent

$$Z = XY \leftrightarrow E[Z] = E[XY] = E[X]E[Y]$$
$$E[g(X, Y)] = E[g_1(X)]E[g_2(Y)]$$

28 Joint Moment

If X, Y discrete:

$$E[X^jY^k] = \sum_i \sum_n x_i^j y_n^k p_{XY}(x_i, y_n)$$

If X, Y jointly continuous:

$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f_{XY}(x, y)dxdy$$

28.1 Correlation

$$E[XY] = E[X^{j=1}Y^{k=1}]$$

If $E[XY] = 0$, then X, Y are orthogonal.

28.2 Central Moment

$$E[(X - E[X])^j \cdot (Y - E[Y])^k]$$

28.2.1 Variance

$$\text{VAR}(X) = E[(X - E[X])^2 \cdot (Y - E[Y])^0]$$

$$\text{VAR}(X) = E[(X - E[X])^2]$$

29 Covariance

$$\text{COV}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[(X - E[X])^1 \cdot (Y - E[Y])^1] = E[XY] - E[X]E[Y]$$

If $E[X] = 0$ and/or $E[Y] = 0$, then

$$\text{COV}(X, Y) = E[XY]$$

29.1 Correlation Coefficient

$$\rho_{XY} = \frac{\text{COV}(X, Y)}{\sigma_X \sigma_Y}, \quad -1 \leq \rho_{XY} \leq 1$$

If X, Y uncorrelated, then

$$\text{COV}(X, Y) = 0, \quad E[XY] = E[X]E[Y], \quad \rho_{XY} = 0$$

If X, Y **independent**, then they are uncorrelated.

29.2 Covariance Properties

$$(X, X) = (X) \quad (X, Y) = (Y, X)$$

$$(\alpha X, Y) = \alpha(X, Y)$$

$$(X + c, Y) = (X, Y)$$

$$(X + Y, Z) = (X, Z) + (Y, Z)$$

30 Conditional Probabilities

30.1 Case 1: X, Y Discrete - Conditional PMF

$$p_Y(y|x) = P[Y = y|X = x] =$$

$$= \frac{P[X = x, Y = y]}{P[X = x]} = \frac{p_{XY}(x, y)}{p_X(x)}$$

$$p_Y(y_k|x_j) = \frac{p_{XY}(x_j, y_k)}{p_X(x_j)} \longrightarrow$$

$$p_{XY}(x_j, y_k) = p_Y(y_k|x_j) \cdot p_X(x_j)$$

$$P[Y \in A|X = x_k] = \sum_{y_j \in A} p_Y(y_j|x_k)$$

$$P[Y \in A] = \sum_{x_k} P[Y \in A|X = x_k]p_X(x_k)$$

30.2 Case 2: X discrete, Y continuous - Conditional PDF

$$F_Y(y|x_k) = \frac{P[Y \leq y, X = x_k]}{P[X = x_k]}, \quad P[X = x_k] > 0$$

$$f_Y(y|x_k) = \frac{d}{dy} F_Y(y|x_k)$$

If X, Y independent,

$$P[Y \in A|X = x_k] = \int_{y \in A} f_Y(y|x_k)dy$$

30.3 Case 3: X, Y continuous - Conditional PDF

$$f_Y(y|x) = \frac{d}{dy} F_Y(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

$$P[Y \in A|X = x] = \int_{y \in A} f_Y(y|x)dy$$

$$P[Y \in A] = \int_{-\infty}^{\infty} P[Y \in A|X = x]f_X(x)dx$$

If X, Y independent,

$$f_Y(y|x) = \frac{f_X(x)f_Y(y)}{f_X(x)} = f_Y(y)$$

30.4 Bayes Rule

$$f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_{XY}(x,y) = f_Y(y|x)f_X(x) = f_X(x|y)f_Y(y)$$

$$f_Y(y|x) = \frac{f_{XY}(x|y)f_Y(y)}{f_X(x)}$$

31 Conditional Expectation

31.1 X,Y Discrete

$$E[Y|x] = \sum_{y_k} p_Y(y_k|x)$$

31.2 X,Y Continuous

$$E[Y|x] = \int_{-\infty}^{\infty} y f_Y(y|x)dy$$

31.3 Law of total Expectation

Since $E[Y|x] = g(X)$, we define $E[g(x)]$

$$E[E[Y|x]] = \int_{-\infty}^{\infty} E[Y|x]f_X(x)dx = E[Y]$$

for any function $h(Y)$, where $E[h(Y)] = E[E[h(Y|x)]]$

$$E[Y^k] = E[E[Y^k|x]]$$

32 Functions of Two RVs

Let $Z = g(X,Y)$ (function of two RVs). Then,

$$F_Z(z) = P[X \in R_z] =_{(x,y)\in R_z} f_{XY}(x,y)dxdy$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)dx$$

If X, Y independent, then

$$f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

33 Transformations of Two RVs

Let $W = (X, Y)$ and $Z_1 = g_1(W)$ and $Z_2 = g_2(W)$

$$F_{z_1,z_2}(z_1, z_2) = P[g_1(W) \leq z_1, g_2(X) \leq z_2]$$

$$F_{z_1,z_2}(z_1, z_2) =_{W:g_k(W)\leq z_k} f_{XY}(x, y)dxdy$$

34 Linear Transformations

$$VW = abcdXY = AXY$$

Assume A is invertible:

$$XY = A^{-1}VW$$

34.1 Joint PDF of Linear Transformation

Let $Z = g(X, Y)$. The vector Z is:

$$Z = AW \qquad Z = VW \quad W = XY$$

The Joint PDF of Z is

$$f_Z(z) = \frac{f_W(A^{-1}z)}{|A|} \quad |A| = \det abcd$$

35 Joint Gaussian RVs

The random variables X, Y are jointly gaussian if:

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho_{XY}^2}} \exp(A)$$

$$A = \frac{-1}{2(1 - \rho_{XY}^2)} \left[\left(\frac{x - m_1}{\sigma_1}\right)^2 - 2\rho_{XY} \left(\frac{x - m_1}{\sigma_1}\right) \left(\frac{y - m_2}{\sigma_2}\right) + \left(\frac{y - m_2}{\sigma_2}\right)^2 \right]$$

35.1 Joint Standard (Normal) Gaussians

If $X\ N(0, 1)$, $Y\ N(0, 1)$, then

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1 - \rho_{XY}^2}} \exp(A)$$

$$A = \frac{1}{2(1 - \rho^2)} \left(x^2 - 2\rho_{XY} \cdot xy + y^2\right)$$

$$f_{XY}(x, y) = g(r) = C \exp\left[\frac{-r^2}{2\sigma^2}\right]$$

35.2 Independence (m=0,σ=1)

If X, Y independent $\leftrightarrow$

$$\text{COV}(X,Y) = 0 \quad \rho_{XY}(x,y) = 0$$

$$f_{XY}(x,y) = \frac{1}{2\pi}\text{exp}\left(-\frac{1}{2}(x^2+y^2)\right)$$

35.3 Independence (m=0)

If X $N(0,1)$, Y $N(0,1)$, then

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2}\text{exp}\left(-\frac{1}{2\sigma^2}(x^2+y^2)\right)$$

35.4 Constant A

If A (exponent of Joint Gaussian) is a constant K

$$K = \left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - + \left(\frac{y-m_2}{\sigma_2}\right)^2\right]$$

$$f_XY(x,y) = C\text{exp}\left[-\frac{1}{2(1-\rho^2)}K\right] = \text{constant}$$

35.5 Major Axis

If X, Y not independent, then the principal axes has

$$\theta = \frac{1}{2}\text{arctan}^{-1}\tan\left(\frac{2\rho_{XY}\sigma_1\sigma_2}{\sigma_1^2-\sigma_2^2}\right)$$

35.6 Conditional PDF

The conditional PDF of X given $Y=y$ is

$$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{2\pi\sigma_1^2\sqrt{1-\rho_{XY}^2}} \cdot \text{exp}\left(\frac{-1}{2(1-\rho_{XY}^2\sigma_1^2)}\left[x-\rho_{XY}\frac{\sigma_1}{\sigma_2}(y-m_2)-m_1\right]^2\right)$$

35.7 Conditional Expectation

$$E[(X-m_1)(Y-m_2)|Y] = (y-m_2)E[X-m_1|Y=y] = (y-m_2)\left(\rho_{XY}\frac{\sigma_1}{\sigma_2}(y-m_2)\right) = \rho_{XY}\frac{\sigma_1}{\sigma_2}(y-m_2)^2$$

35.8 Covariance

$\text{COV}(X, Y) = E[(X - m_1)(Y - m_2)] = E[E[(X - m_1)(Y - m_2)|Y]] = \rho_{XY}\sigma_1\sigma_2$

36 Sum of RVs

Let $X_1, X_2, ..., X_n$ be a sequence of RVs, with

$$S_n = X_1 + X_2 + +X_n$$

36.1 Mean and Variance of Sum of RVs

$$E[X_1 + X_2 + +X_n] = E[X_1] + E[X_2] + +E[X_n]$$

$$(X_1 + +X_n) = \sum_{k=1}^{n}(X_k) + \sum_{j=1}\sum_{k=1}(X_j, X_k), \quad j \neq k$$

If $X_1, X_2, ..., X_n$ independent, then

$$(X_1 + +X_n) = (X_1) + +(X_n)$$

36.2 PDF of Sums of Independent RVs

Let $X_1, X_2, ..., X_n$ independent, then

$$\phi_{S_n}(\omega) = E[e^{j\omega S_n}] = E[e^{j\omega(X_1+X_2++X_n)}]E[e^{j\omega X_1}]E[e^{j\omega X_n}] = \phi_{X_1}(\omega)\phi_{X_n}(\omega)$$

$$f_{S_n} =^{-1} [\phi_{X_1}(\omega)\phi_{X_n}(\omega)]$$

37 Independent Identically Distributed RVs (iid)

If $X_1, X_2, ..., X_n$ iid RVs, with

$$E[X_j] = m_x \quad (X_j) = \sigma_x^2 \quad \text{for } j = 1, ..., n$$

37.1 Mean and Variance of iid RVs

$$E[S_n] = E[X_1] + +E[X_n] = n \cdot m_x$$

$$(S_n) = n \cdot (X_j) = n\sigma_x^2$$

37.2 PDF of iid RVs

$$\phi_{X_k}(\omega) = \phi_X(\omega)\,, k = 1, ..., n \leftrightarrow \phi_{S_N}(\omega) = [\phi_X(\omega)]^n$$

$$f_{S_n} =^{-1} (\phi_{S_n}(\omega)) =^{-1} (\phi_X(\omega)^n)$$

38 Sample Mean

$$M_n = \frac{1}{n}\sum_{j=1}^{n} X_j$$

38.1 Expected Value and Varianceof Sample Mean

$$E[M_n] = E\left[\frac{1}{n}\sum_{j=1}^{n} X_j\right] \rightarrow E[M_n] = \frac{1}{n}\sum_{j=1}^{n} E[X_j]$$

$$(M_n) = E\left[(M_n - E[M_n])^2\right] = (S_n)/n^2$$

if $X_1, , X_n$ iid RVs:

$$E[M_n] = m_x \leftrightarrow E[S_n] = n \cdot m_x$$

$$(S_n) = n\sigma^2 \leftrightarrow (M_n) = \frac{\sigma^2}{n}$$

38.2 Sample Mean Chebyshev Bound

$$P[|Z - E[Z]| \geq \epsilon] \leq \frac{(Z)}{\epsilon^2} \quad , \epsilon > 0$$

$$P[|M_n - m_x| \geq \epsilon] \leq \frac{\sigma^2}{n\epsilon^2}$$

$$P[|M_n - m_x| < \epsilon] \geq 1 - \frac{\sigma^2}{n\epsilon^2}$$

39 Laws of Large Numbers

$$\text{Weak Law :} \quad \lim_{n\to\infty} P[|M_n - m_x| < \epsilon] = 1$$

$$\text{Strong Law :} \quad P\left[\lim_{n\to\infty} M_n = m_x\right] = 1$$

40 Central Limit Theorem

Let $S_n = X_1, X_2, ..., X_n$ iid RVs

$$\lim_{n\to\infty} P[Z_n \leq z] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{z} e^{-x^2/2} dx$$