# ECE318 Course Notes

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# 1 Geometric Optics

## 1.1 Optical Path Length (OPL)

$$OPL = \int_A^B n(s) \cdot ds$$

## 1.2 Interfaces

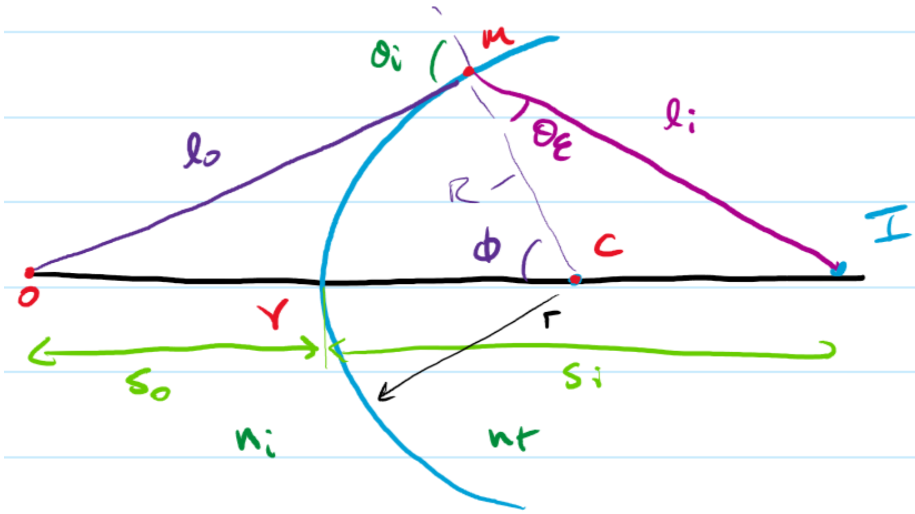
| | |
|---|---|
| Reflection | $\theta_i = \theta_r$ |
| Refraction (Snell's Law) | $n_i \sin(\theta_i) = n_t \sin(\theta_t)$ |

## 1.3 Fermat's Principle

Light traverses shortest OPL route, $\frac{dOPL(x)}{dx} = 0$

### 1.4 Optical Imaging Systems

### 1.5 Spherical Refractive Surface




#### 1.5.1 Paraxial Approximation

$$\frac{S_o}{n_i(S_o + R)} = \frac{S_i}{n_t(S_i - R)}$$

#### 1.5.2 Gaussian Formula

$$\frac{n_i}{S_o} + \frac{n_t}{S_i} = \frac{n_t - n_i}{R}$$

### 1.6 Thin Lens

#### 1.6.1 Lensmakers Formula

$$\frac{1}{f} = (n_t - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{n_l - n_1}{n_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

## 2 Wave Optics 1

Light is a harmonic (single-frequency, monochromatic) wave.

## 2.1 Maxwell Equations

| Integral Form | Differential Form |
| --- | --- |
| $\oint_C E \cdot dl = -\iint_s \frac{\partial B}{\partial t}$ | $\nabla \times E = -\frac{\partial B}{\partial t}$ |
| $\oint_C H \cdot dl = \iint_s \left(\frac{\partial D}{\partial t} + J\right) = I_{enc}$ | $\nabla \times H = \frac{\partial D}{\partial t} + J$ |
| $\oint_S D \cdot ds = \iiint \rho dV = Q$ | $\nabla \cdot D = \rho$ |
| $\oint_S B \cdot ds = 0$ | $\nabla \cdot B = 0$ |

## 2.2 Materials

| | |
| --- | --- |
| Source free | $J = 0, \rho = 0$ |
| Linear | $\|P\| \propto \|E\|, \|M\| \propto \|H\|$ |
| Isotropic | $\epsilon_r \mu_r$ scalars |

As a result,

$$D = \epsilon_0 \epsilon_r E = \epsilon E \quad B = \mu_0 \mu_r H = \mu H$$

## 2.3 Harmonic Plane Waves

$$E(r,t) = E_0 \cos(\vec{k} \cdot \vec{r} \pm \omega t + \phi)$$

$$E(r,t) = E_0 \exp[i(\vec{k} \cdot \vec{r} \pm \omega t + \phi)]$$

$\vec{E}$ a real quantity (take real part of $E(r,t)$

## 2.4 Wave Properties

Temporal period and frequency:

$$T = \frac{2\pi}{\omega} \quad v = \frac{\omega}{2\pi}$$

Spatial period and frequency:

$$\lambda = \frac{2\pi}{|k|} \quad f = \frac{|k|}{2\pi}$$

Speed of propagation:

$$v_\phi = \frac{spatial\ period}{temporal\ period} = \frac{\lambda}{2\pi/\omega} = \frac{\omega}{|k|} = \frac{1}{\sqrt{\mu\epsilon}}$$

Perpendicularity ($E, H, k$ right hand triplet)

$$k \perp \vec{E},\ k \perp \vec{H} \qquad k \times E = \mu\omega H$$

If $E$, $H$ in phase,

$$\frac{|E|}{|H|} = \sqrt{\frac{\mu}{\epsilon}}$$

## 2.5 Other Properties

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

$$k = \frac{\omega}{c} n \qquad \frac{\omega}{k} = v = \frac{c}{n} = \frac{\lambda}{T}$$

## 2.6 Poynting Vector & Power

$$\vec{S} = \vec{E} \times \vec{H}$$

Hard to calculate $S$, since only (time-averaged) power can be measured ($S$ is power density)

## 2.7 Irradiance

$$I = \langle S \rangle_T = \langle |E \times H| \rangle_T = \langle E_0 H_0 \cos^2(k \cdot r - \omega t + \phi) \rangle_T$$

$$I = \frac{1}{2} E_0 H_0 = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} E_0^2 = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} H_0^2$$

$$I \propto |E|^2$$

## 2.8 Polarization

Prop in $\hat{z}$, Lin. pol in $y$:

$$E(r,t) = E_0 \hat{y} \exp[i(kz - \omega t + \phi)]$$

Lin. pol in $x, y$:

$$E(r,t) = (E_{x0} \hat{x} + E_{y0} \hat{y}) \exp[i(kz - \omega t + \phi)]$$

Circ. pol:

$$E(r,t) = (E_0 \hat{x} + E_0 e^{i\frac{\pi}{2}} \hat{y}) \exp[i(kz - \omega t + \phi)]$$

Ellip. pol:

$$E(r,t) = (E_{x0} \hat{x} + E_{y0} e^{i\phi} \hat{y}) \exp[i(kz - \omega t + \phi)]$$

In general:

$$E(r,t) = (E_{x0} e^{i\phi_x} \hat{x} + E_{y0} e^{i\phi_y} \hat{y}) \exp[i(kz - \omega t + \phi)]$$

- Lin: $\phi_y - \phi_x = m\pi$
- Circ: $\phi_y - \phi_x = \frac{\pi}{2} + m\pi$
- Ellip: $E_{x0} \neq E_{y0}$ and $\phi_y - \phi_x \neq m\pi$
- Ellip: $E_{x0} = E_{y0}$ and $\phi_y - \phi_x \neq m\pi + \pi/2, m\pi$

## 2.9 Jones Vectors

$$J = \begin{bmatrix} E_{x0}e^{i\phi_x} \\ E_{y0}e^{i\phi_y} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{E_{y0}}{E_{x0}} e^{i(\phi_y - \phi_x)} \end{bmatrix}$$

### 2.9.1 Jones Vector Properties

- Normalized: $|J| = 1,\ J^* \cdot J = 1$
- $J_1, J_2$ orthogonal if $J_1^* \cdot J_2 = 0$
- Linearity: $J = \alpha J_1 + \beta J_2$

### 2.9.2 Jones Vector Examples

Lin pol wrt $x$:

$$J = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Lin pol $\theta$ degrees wrt $x$:

$$J = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Left hand circ pol:

$$J = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Right hand circ pol:

$$J = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

## 2.10 Rotation of Polarization

$$J' = R(\psi)J = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}$$

Circ polarizations are rotation-invariant (although have an added phase)

## 2.11 Malus's Law

Linearly polarized light:

$$I_{in} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} E_{in}^2$$

After passing through a lin polarizer:

$$I_{out} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} E_{out}^2 = \cos^2\theta I_{in}$$

Cirularly polarized light:

$$I_{in} = \sqrt{\frac{\epsilon}{\mu}} E_{in}^2$$

After passing through a lin polarizer:

$$I_{out} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}E_{out}^2 = \frac{1}{2}I_{in}$$

Elliptically polarized light:

$$I_{in} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}(E_x^2 + E_y^2)$$

After passing through a lin polarizer:

$$I_{out} = \frac{\cos^2\theta E_x^2 + \sin^2\theta E_y^2}{E_x^2 + E_y^2} I_{in}$$

## 2.12 Jones Matrices

$$J_o = T \cdot J_i = \begin{bmatrix} a & b \\ c & d \end{bmatrix} J_i$$

### 2.12.1 Eigenvectors

$$TJ = \alpha J$$

Eigenvectors of a 2x2 $T$ matrix are the independent polarization states. Light with polarization state corresponding to eigenvector goes through $T$ unchanged.

### 2.12.2 Jones Matrix Examples

TA @ $\theta$ wrt x axis:

$$T = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

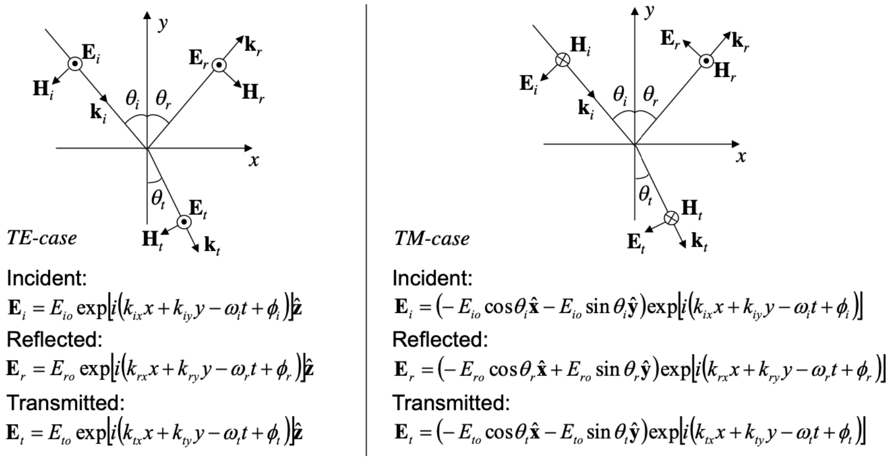
## 2.13 Wave Plates

$$T = \begin{bmatrix} e^{i\phi_s} & 0 \\ 0 & e^{i\phi_f} \end{bmatrix} = e^{\phi_f} \begin{bmatrix} e^{i\Delta\phi} & 0 \\ 0 & 1 \end{bmatrix}$$

Where $\Delta\phi = \phi_s - \phi_f$

- QWP: $\Delta\phi = m\pi + \pi/2$
- HWP: $\Delta\phi = 2m\pi + \pi$

## 2.14 Reflection and Refraction at Interfaces




### 2.14.1 Phase Matching at Boundary

$$\omega_i = \omega_r = \omega_t = \omega$$

$$k_{ix} = k_{rx} = k_{tx} \rightarrow \frac{\omega}{c} n_i \sin\theta_i = \frac{\omega}{c} n_r \sin\theta_r = \frac{\omega}{c} n_t \sin\theta_t$$

Since $n_i = n_r$,

$$\theta_i = \theta_r \rightarrow n_i \sin\theta_i = n_r \sin\theta_r$$

And also

$$\phi_i = \phi_r = \phi_t = \phi$$

## 2.15 Fresnel Coefficients

$$r_{TE} = \left(\frac{E_r}{E_i}\right)_{TE} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_{TE} = \left(\frac{E_t}{E_i}\right)_{TE} = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$r_{TM} = \left(\frac{E_r}{E_i}\right)_{TM} = \frac{-n_t\cos\theta_i - n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t}$$

$$t_{TM} = \left(\frac{E_t}{E_i}\right)_{TM} = \frac{2n_i\cos\theta_i}{n_t\cos\theta_i + n_i\cos\theta_t}$$

## 2.16 Reflections

Internal reflections: $n_i < n_t$ External reflections: $n_i > n_t$

## 2.17 Reflectance and Transmittance

$$R \equiv \frac{P_r}{P_i} = \frac{I_r A \cos\theta_i}{I_i A \cos\theta_i} = r^2$$

$$T \equiv \frac{P_t}{P_i} = \frac{I_t A \cos\theta_t}{I_i A \cos\theta_i} = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} t^2$$

Energy conservation:

$$R + T = 1$$

### 2.17.1 Normal Incidence

If $\theta_i = 0$,

$$r_{TE} = r_{TM} = \frac{n_i - n_t}{n_i + n_t} \to R_{TE} = R_{TM} = \left(\frac{n_i - n_t}{n_i + n_t}\right)^2$$

Plane of incidence is not unique. No polarization dependence

### 2.17.2 Brewster's Angle

At Brewster's angle ($\theta_p$), TM polarized light does not reflect ($r_{TM} = 0$).

$$n_t \cos\theta_{ip} = n_i \cos\theta_{tp} \quad n_i \sin\theta_{ip} = n_t \sin\theta_{tp}$$

$$\tan\theta_{ip} = \frac{n_t}{n_i}$$

$$\theta_{ip} + \theta_{tp} = \frac{\pi}{2}$$

## 2.18 Total Internal Reflectance

Two conditions:

- $n_i > n_t$
- $\theta_i > \theta_c$

## 2.19 Critical Angle

Incident angle for internal reflection, when $\theta_t = \frac{\pi}{2}$:

$$\theta_c = \sin^{-1} \frac{n_t}{n_i}$$

### 2.20 Evanescent Waves

The transmitted wave in TIR case is evanscent:

$$|k_t| = \frac{\omega}{c} n_t \rightarrow k_{tx} = k_{ix} = k_i \sin\theta_i = \frac{\omega}{c} n_i \sin\theta_i$$

$$k_{ty} = \sqrt{k_t^2 - k_{tx}^2} = \pm i \frac{\omega}{c} n_t \sqrt{\left(\frac{n_i \sin\theta_i}{n_t}\right)^2 - 1}$$

$$\beta = \frac{\omega}{c} n_t \sqrt{\left(\frac{n_i \sin\theta_i}{n_t}\right)^2 - 1}$$

$$E_t = E_{t0} e^{\beta y} \exp[i(\ldots)]$$

### 2.21 Penetration Depths

Field/amplitude penetration depth:

$$E(y = \frac{1}{\beta}) = \frac{1}{e} E(y = 0)$$

Intensity penetration depth

$$I(y = \frac{1}{2\beta}) = \frac{1}{e} I(y = 0)$$

### 2.22 Complex Fresnel Coefficients

Fresnel coeffs can be applied to TIR:

$$r_{TE} = e^{i\phi_{TE}} \qquad r_{TM} = e^{i\phi_{TM}}$$

$$\phi_{TE} = -2\tan^{-1}\left(\frac{n_t \sqrt{(n_i \sin\theta_i / n_t)^2}}{n_i \cos\theta_i}\right)$$

$$\phi_{TM} = -2\tan^{-1}\left(\frac{n_i \sqrt{(n_i \sin\theta_i / n_t)^2}}{n_t \cos\theta_i}\right)$$

# 3 Wave Optics 2

## 3.1 Interference

Given two waves

$$E_1 = E_{1o} \cos(\vec{k_1} \cdot \vec{r} - \omega_1 t + \phi_1)$$

$$E_2 = E_{2o} \cos(\vec{k_2} \cdot \vec{r} - \omega_1 t + \phi_1)$$

$$I_1 = \langle E_1 \times H_1 \rangle = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_{1o}|^2$$

$$I_2 = \langle E_2 \times H_2 \rangle = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_{2o}|^2$$

Superposition $E = E_1 + E_2$ yields:

$$I = \sqrt{\frac{\epsilon}{\mu}} = I_1 + I_2 + 2\sqrt{\frac{\epsilon}{\mu}}\langle E_1 \cdot E_2 \rangle$$

interference term: $2\sqrt{\frac{\epsilon}{\mu}}\langle E_1 \cdot E_2 \rangle$

### 3.1.1 Interference Term

$$2\sqrt{\frac{\epsilon}{\mu}}\langle E_1 \cdot E_2 \rangle =$$

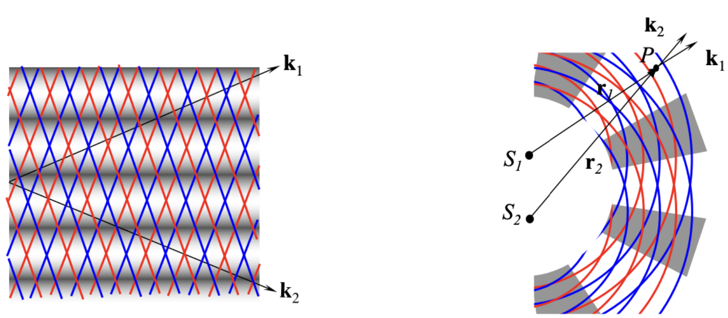
$$\sqrt{\frac{\epsilon}{\mu}}(E_{1o} \cdot E_{2o})\langle \cos[(k_1 - k_2) \cdot r - (\omega_1 - \omega_2)t + (\phi_1 - \phi_2)] \rangle$$

## 3.2 Spherical Interference

In spherical coordinates $(r, \theta, \phi)$:

$$\sqrt{\frac{\epsilon}{\mu}}\left(\frac{E_{1o}}{r_1} \cdot \frac{E_{2o}}{r_2}\right)\langle \cos[(k_1 r_1 - k_2 r_2) - (\omega_1 - \omega_2)t + (\phi_1 - \phi_2)] \rangle$$

## 3.3 Visualization of Interference




*The red lines represent the equal phase lines of $2m\pi$, and the blue lines are $(2m+1)\pi$ phase lines. When red lines meet red lines (or blue lines meet blue lines), the two waves are in phase, and the resulting intensity is the highest (the bright fringe in the background). When red lines meet blue lines, the two waves are $\pi$ out-of-phase, and the resulting intensity is the lowest (dark fringe). For the plane waves (left figure), the two* **k** *vectors are in the same co-ordinate system. For the spherical waves (right figure), the two* **k** *vectors (or* **r** *vectors) for a given point of interest, P, are in separate co-ordinate systems.* **k** *and* **r** *are therefore always collinear. Hence,* $\mathbf{k} \cdot \mathbf{r} = kr$

### 3.4 Conditions for Non-interference

1. orthogonal polarization: $E_{1o} \perp E_{2o}$
2. $\omega_1 \neq \omega_2$ s.t. time avg of fast varying cos is 0
3. $\phi_1 - \phi_2$ varies with time randomly

### 3.5 Conditions for Interference

1. $\omega_1 = \omega_2$
2. $E_{1o}$ not $\perp$ $E_{2o}$
3. $\phi_1 - \phi_2$ not time varying

### 3.6 Relative Phase and OPL Diff

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

where $\delta$ is the relative phase between two interfering waves

$$\delta = (k_1 - k_2) \cdot r + (\phi_1 - \phi_2) \quad \text{plane waves}$$

$$\delta = (r_1 - r_2) + (\phi_1 - \phi_2) \quad \text{plane waves}$$

### 3.7 Fringes

- bright: $\delta = 2m\pi \rightarrow I = I_1 + I_2 + 2\sqrt{I_1 I_2}$
- dark: $\delta = (2m+1)\pi \rightarrow I = I_1 + I_2 - 2\sqrt{I_1 I_2}$

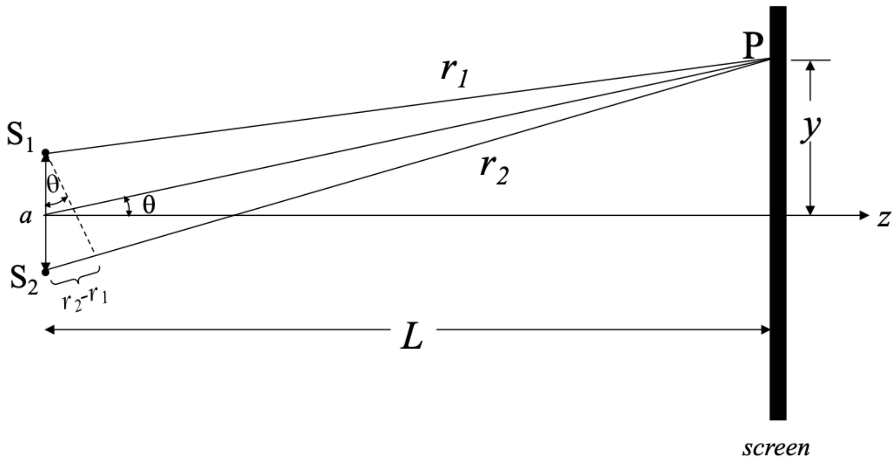
### 3.8 Equal Intensity Interference

If $I_1 = I_2 = I$

$$I = I_1 + I_1 + 2\sqrt{I_1 I_1}\cos\delta = 2I_1(1 + \cos\delta) =$$

$$= 4I_1 \cos^2\left(\frac{\delta}{2}\right)$$

## 3.9 Interference of two point sources




$$\Delta OPL = r_2 - r_1$$
$$\delta = 2\pi\frac{\Delta OPL}{\lambda} + (\phi_1 - \phi_2) = 2\pi\frac{r_2 - r_1}{\lambda_0}$$

Small angle approx: $r_2 - r_1 \approx a\theta \approx a\frac{y}{L}$

$$\delta = 2\pi\frac{ay}{\lambda_0 L}$$

- bright: $\delta = 2\pi\frac{ay_{bt}}{\lambda_0 L} = 2m\pi$
- dark: $\delta = 2\pi\frac{ay_{dk}}{\lambda_0 L} = 2m\pi + \pi$

or

- bright: $y_{bt} = m\frac{\lambda_0 L}{a}$
- dark: $y_{dk} = \left(m + \frac{1}{2}\right)\frac{\lambda_0 L}{a}$

Fringe spacing:

$$\Delta y_{fringe} = \frac{\lambda_0 L}{a}$$

## 3.10 Coherence

For two waves to have long-lasting interference, they must have a fixed phase relationship:

$$\phi_1 - \phi_2 \text{ must not be time varying}$$

## 3.11 Practical Light Sources

Practical sources are not monochromatic or point sources

## 3.12 Temporal Coherence

$\tau_c$ is the average duration of wave trains. $l_c$: longitudinal coherence length.

$$l_c = c\tau_c$$

Long-lasting interference cannot be observed if $\Delta OPL > l_c$. Coherence condition: $\Delta OPL < l_c$

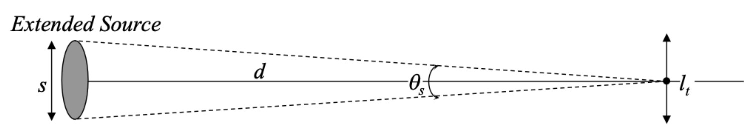
## 3.13 Spatial Coherence

$l_t$: spatial coherence length

$$l_t \approx \frac{\lambda}{\theta_s} = \frac{\lambda d}{s} \quad (11\text{-}53)$$

$\theta_s$ being the angle subtended by the source, viewed from the point of interest (see below). For circular sources,

$$l_t = 1.22\frac{\lambda}{\theta_s} = 1.22\frac{\lambda d}{s} \quad (11\text{-}54)$$

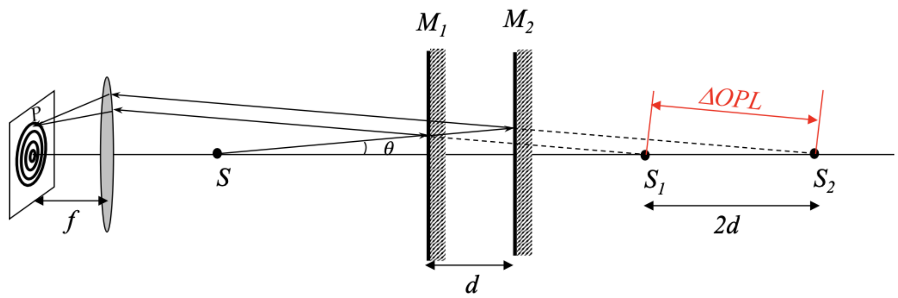



## 3.14 Spectral Linewidth

$$\Delta v = \frac{1}{\tau_c}$$

$$\frac{\Delta v}{v} = \frac{\Delta \lambda}{\lambda}$$

## 3.15 Michelson Interferometer




$$\Delta OPL = 2d\cos\theta$$

Phase difference is thus:

$$\delta = k\cdot\Delta OPL + \pi = \frac{4\pi d\cos\theta}{\lambda} + \pi$$

Center of a bright fringe occurs at

$$\delta_{bt_m} = 2m\pi \quad or \quad 2d\cos\theta_{bt_m} = (m - \frac{1}{2})\lambda$$

### 3.15.1 Michelson Fringe Radii

The $p$th bright fringe in center ($\theta = 0$):

$$2d = (p - \frac{1}{2})\lambda$$

The $m$th bright fringe from center $m = p - N$:

$$2d\cos\theta_{bt_m} = (m - \frac{1}{2})\lambda$$

Small incident angles:

$$\cos \approx 1 - \frac{\theta^2}{2}$$

$$\theta_{bt_m}^2 \approx \frac{2d - (m - \frac{1}{2}\lambda)}{d}$$

$$\theta_{bt_m} \approx \sqrt{\frac{(p-m)\lambda}{d}} = \sqrt{\frac{N\lambda}{d}}$$

Radii of bright fringes:

$$r_{bt_m} = f\theta_{bt_m} = f\sqrt{\frac{N\lambda}{d}}$$

Where $f$ is the focal length of the lens

### 3.15.2 Fringe Separation for Michelson

$$\Delta r_{bt_N} = f(\theta_{bt_{N+1}} - \theta_{bt_N}) = f\sqrt{\frac{N\lambda}{d}}(\sqrt{N+1} - \sqrt{N})$$

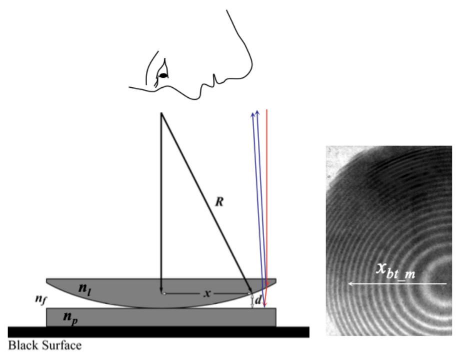
Fringe spacing is not uniform (decreases from center to edge). Spacing between $N$ and $N+1$ is proportional to $\lambda$, inversely proportional to $\sqrt{d}$

### 3.15.3 Fringe Distortion

After a path length of $\Delta d$, the fringe distortion is

$$\Delta m\lambda = 2\Delta d$$

## 3.16 Newton's Rings




Let $d$ be the thickness of the space between the two dielectrics, the relationship between $x$, $d$ and $R$ is:

$$x^2 + (R-d)^2 = R^2 \qquad (12\text{-}12)$$

For a small $d$, we can ignore the 2nd order term $d^2$, and write:

$$x = \sqrt{2Rd} \qquad (12\text{-}13)$$

Furthermore, ignoring the small angle tilt in the reflected beams due to refraction, $2n_f d$ is the $\Delta OPL$ between the two interfering beams.

May be additional $\pi$ phase between two beams. Assume $n_f < n_l$, $n_f < n_p$
Small angle approx:

$$\Delta OPL \approx 2d$$

$$\delta = 2d = \frac{2\pi md}{\lambda}\cos\theta + \pi$$

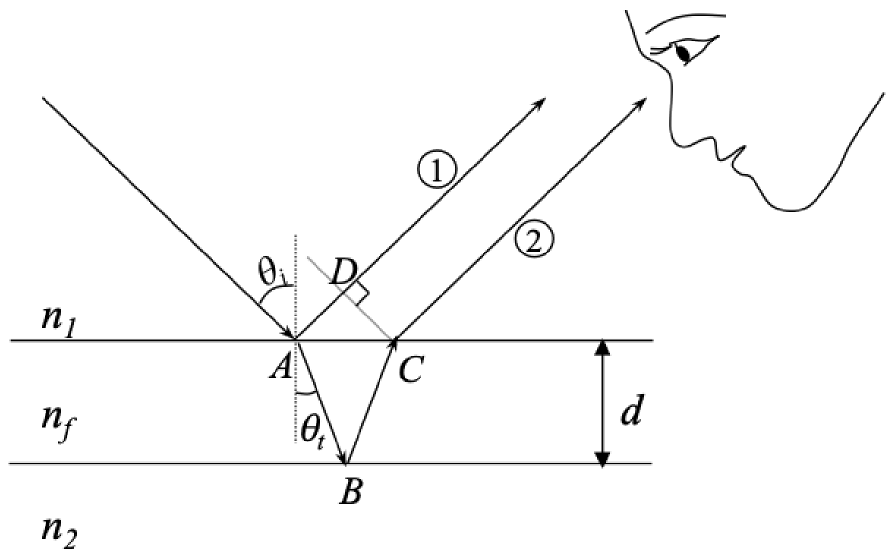
bright fringes appear at $\delta = 2m\pi$

### 3.16.1 Newtons Rings Fringe Radii

radius of bright fringes:

$$r_{bt_m} = \sqrt{\frac{R\lambda_0}{n_f}(m - \frac{1}{2})}$$

### 3.17 Thin-Film Interference




$$\Delta OPL = \frac{2n_f d}{\cos\theta_t} - 2n_1 d \tan\theta_t \sin\theta_i$$
$$\Delta OPL = 2n_f d \cos\theta_t$$

$$\delta = 2\pi \frac{2n_f d \cos\theta_t}{\lambda_0} + \pi$$

bright fringes appear at $\delta = 2m\pi$

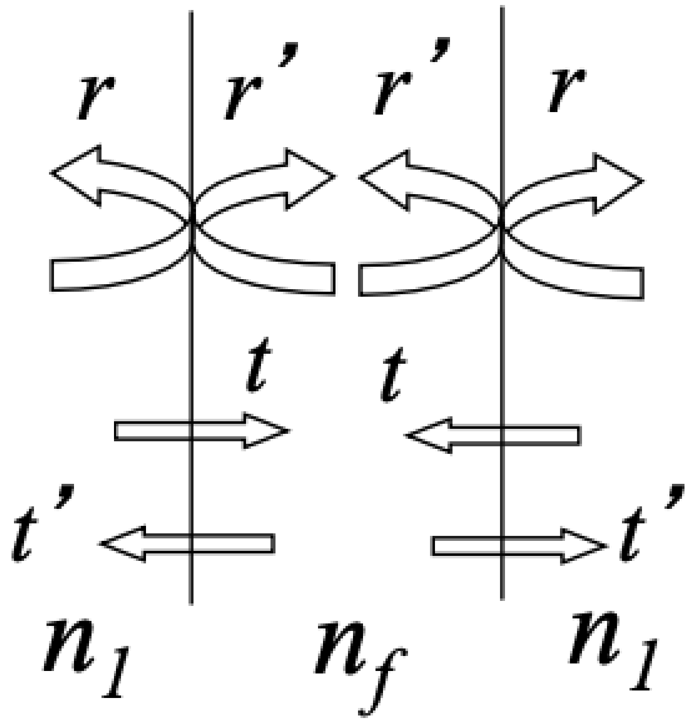
$$2n_f d \cos\theta_t = (m - \frac{1}{2})\lambda_0$$

### 3.18 Fabry Perot Interference

$$\delta = 2\pi \frac{2n_f d}{\lambda_0} \cos\theta_t$$

#### 3.18.1 Fresnel Coefficients at thin-film

Drop $TE$, $TM$ subscript at near incidence condition.




$$r = -r'$$
$$r^2 = (r')^2 = R$$
$$tt' = T = 1 - R \qquad T \neq t^2$$

### 3.19 Coefficient of Finesse

$$F \equiv \frac{4R}{(1-R)^2}$$

### 3.20 Transmittance of Fabry Perot

$$T_{FP} = \frac{1}{1 + F\sin^2(\delta/2)}$$
$$\delta = 2\pi \frac{2n_f d\cos\theta_t}{\lambda_0}$$

## 3.21 Reflectance of Thin-Film

$$R_{TF} = 1 - T_{FP}$$

## 3.22 Airy Function

When $\delta = (2m+1)\pi$

$$T_{FP,min} = \frac{1}{1+F}$$

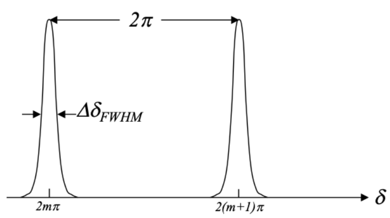
When $\delta = 2m\pi$

$$T_{FP,max} = 1$$

## 3.23 Finesse

$$\mathbb{F} = \frac{\text{Fringe Spacing}}{\text{FWHM Fringe Width at Resonance}}$$

FWHM fringe width $= \Delta\delta_{FWHM}$




$$\mathscr{F} = \frac{2\pi}{\Delta\delta_{FWHM}} \qquad (12\text{-}50)$$

We shall proceed to find $\Delta\delta_{FWHM}$ using Eq (12-39).

### 3.23.1 Full Width at Half Maximum

Fringe width when transmittance drops to half of peak value

$$\mathbb{F} = \frac{2\pi}{\Delta\delta_{FWHM}} = \frac{\pi\sqrt{F}}{2}$$

## 3.24 Resolving Power

$$\Delta\lambda_{RP} = 2n_f d \sin\theta_{tm} \Delta\theta_{FWHM}/m$$

$$\mathbb{R} = \frac{\lambda_0}{\Delta\lambda_{RP}}$$

Since

$$\delta = 2\pi \frac{2n_f d \cos\theta_t}{\lambda_0}$$

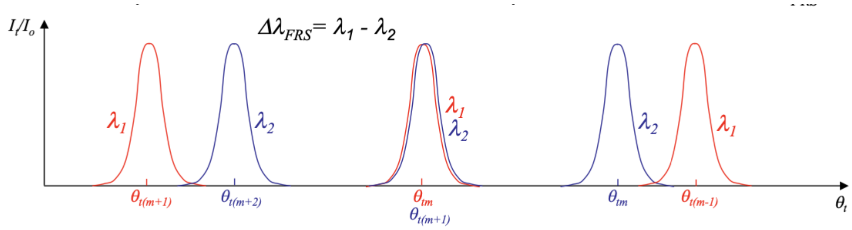
$$\Delta\lambda_{RP} = \frac{\lambda_0}{m\mathbb{F}}$$

For the FP, resolving power is defined as:

$$\mathbb{R} = \frac{\lambda_0}{\Delta\lambda_{RP}} = m\mathbb{F}$$

$$\mathbb{R} = m\mathbb{F} = \frac{2n_f d}{\lambda_0}\mathbb{F}$$

### 3.25 Free Spectral Range




$m$th order bright fringe of $\lambda_1$ overlaps with $m+1$ bright fringe of $\lambda_2$.

$$2n_f d \cos\theta_{tm}|_{\lambda=\lambda_1} = m\lambda_1$$

$$2n_f d \cos\theta_{t(m+1)}|_{\lambda=\lambda_2} = (m+1)\lambda_2$$

FSR is the largest range in a given order that doesnt overlap same range in another order. Also the largest unambiguous measurement range.

$$\Delta\lambda_{FSR} = \lambda_1 - \lambda_2 = \frac{\lambda_2}{m}$$

FSR range reduces as $m$ increases

$$\Delta\lambda_{FSR} \approx \frac{\lambda_0^2}{2n_f d}$$

$$\frac{\Delta v_{FSR}}{v} = \frac{\Delta\lambda_{FSR}}{\lambda_0} \approx \frac{\lambda_0}{2n_f d} = \frac{c/v}{2n_f d}$$

$$\Delta v_{FSR} = \frac{c}{2n_f d}$$

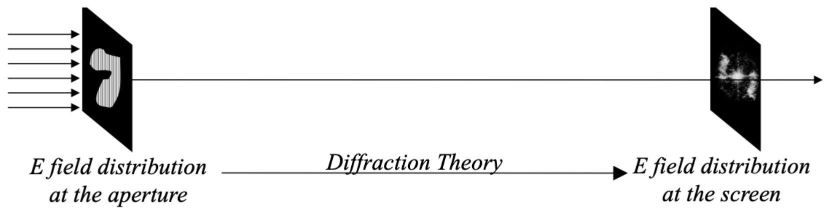
## 4 Wave Optics 3

### 4.1 Fourier Transforms

$$f(t) \leftrightarrow \int f(t)e^{-j\omega t}dt$$

$$f(x,y) \leftrightarrow \int f(x,y)e^{-jk_x x}e^{-jk_y y}dt$$

## 4.2 Diffraction




Define

- $E_o$: field distribution at aperture
- $E_i$: field distribution at screen

$$E_i(x_i, y_i) = \iint_{-\infty}^{+\infty} C \frac{E_o(x_o, y_o)}{r} e^{ikr} dx_o dy_o$$

Where

- $r$: distance between point source $(x_o, y_o)$ and point on screen
- $C$: Proportionality constant

### 4.2.1 Approximations for $r$

In rect coordinates,

$$r = \sqrt{z_i^2 + (x_i - x_o)^2 + (y_i - y_o)^2}$$

Amplitude approx:

$$r \approx z_i$$

Phase approx:

$$r = z_i \sqrt{1 + \frac{(x_i - x_o)^2 + (y_i - y_o)^2}{z_i^2}}$$

$$r \approx z_i + \frac{x_i^2 + y_i^2}{2z_i} - \frac{x_i x_o + y_i y_o}{z_i} + \frac{x_o^2 + y_o^2}{2z_i}$$

## 4.3 Fraunhofer Region (Far Field)

$$r = z_i \sqrt{1 + \frac{(x_i - x_o)^2 + (y_i - y_o)^2}{z_i^2}}$$

$$r \approx z_i + \frac{x_i^2 + y_i^2}{2z_i} - \frac{x_i x_o + y_i y_o}{z_i}$$

## 4.4 Fresnel Region (Near Field)

$$r = z_i\sqrt{1 + \frac{(x_i - x_o)^2 + (y_i - y_o)^2}{z_i^2}}$$

$$r \approx z_i + \frac{x_i^2 + y_i^2}{2z_i} - \frac{x_i x_o + y_i y_o}{z_i} + \frac{x_o^2 + y_o^2}{2z_i}$$

## 4.5 Far Field Condition

When can last term in $r$ approx be negligible?

$$k\frac{x_o^2 + y_o^2}{2z_i} << 2\pi$$

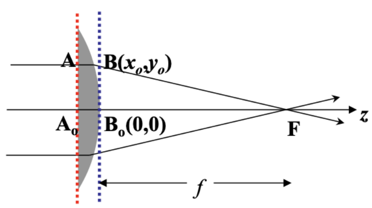
or

$$z_i >> k\frac{x_o^2 + y_o^2}{2\lambda}$$

$$\sqrt{x_o^2 + y_o^2} << \sqrt{\lambda z_i}$$

### 4.5.1 Far Field Compensation (with Lens)

Another way to compensate for quadratic term in far-field condition is to use a lens




Consider a plane wave incident on a convex lens and focuses to the point **F**. As the incident wave is a plane wave, the phase anywhere on the transverse plane just before the lens (indicated by the red line) is the same. From geometric optics, we also know that the OPL for the ray going from **A** to **F** is the same as the OPL from $\mathbf{A_o}$ to **F**.

Therefore, the phases for all the rays at **F** are the same, which we label as $\phi_F$. Let's calculate the phases on the transverse plane after the lens (blue line) at $\mathbf{B_o}(0,0)$, $\phi_{Bo,}$ and $\mathbf{B}(x_o, y_o)$, $\phi_B$:

Since $OPL_{B_oF} = f$ and $OPL_{BF} = \sqrt{f^2 + x_o^2 + y_o^2} \approx f + \left(x_o^2 + y_o^2\right)/2f$, therefore,

$\phi_{B_oF} = \phi_F - k\,OPL_{B_oF}$ and $\phi_{BF} = \phi_F - k\,OPL_{BF}$. The phase difference being

$$\phi_{BF} - \phi_{B_oF} = -k\left(OPL_{BF} - OPL_{B_oF}\right) = -k\left(x_0^2 + y_0^2\right)/2f \qquad \text{(13-7)}$$

## 4.6 Spatial Fourier Transform

$$E_i(x_i, y_i) = \iint_{-\infty}^{+\infty} C \frac{E_o(x_o, y_o)}{z_i} \exp\left[ik\left(z_i + \frac{x_i^2 + y_i^2}{2z_i} - \frac{x_i x_o + y_i y_o}{z_i}\right)\right] dx_o dy_o$$

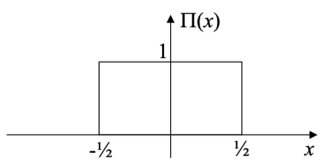
$$= \frac{C}{z_i} e^{ik\left(z_i + \frac{x_i^2 + y_i^2}{2z_i}\right)} \underbrace{\iint_{-\infty}^{+\infty} E_o(x_o, y_o) \exp\left[-ik\left(\frac{x_i}{z_i} x_o + \frac{y_i}{z_i} y_o\right)\right] dx_o dy_o}_{\text{2D spatial Fourier Transform}} \qquad (13\text{-}8)$$

The integral above is a 2D spatial Fourier Transform with

$$\begin{cases} f_x = \dfrac{k_x}{2\pi} = \dfrac{x_i}{\lambda z_i} = \dfrac{k \sin\theta}{2\pi} \\ f_y = \dfrac{k_y}{2\pi} = \dfrac{y_i}{\lambda z_i} = \dfrac{k \sin\varphi}{2\pi} \end{cases} \qquad (13\text{-}9)$$

## 4.7 Rectangle Function

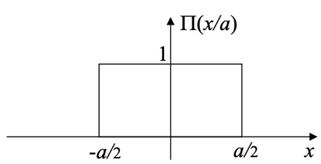
(1) Rectangle function




$$\Pi(x) \equiv \begin{cases} 1 & |x| \le 1/2 \\ 0 & |x| > 1/2 \end{cases} \qquad (13\text{-}10)$$

$$\mathcal{F}\{\Pi(x)\} = \frac{\sin(\pi f)}{\pi f} \equiv \operatorname{sinc} f \qquad (13\text{-}11)$$

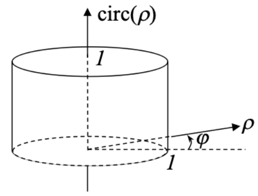
*Rectangle function with scaling:*




$$\mathcal{F}\left\{\Pi\left(\frac{x}{a}\right)\right\} = a \operatorname{sinc}(af) \qquad (13\text{-}12)$$

## 4.8 Circle Function

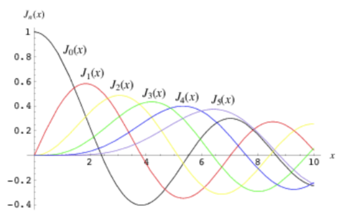
(2) Circle function




$$\mathrm{circ}(\rho) \equiv \begin{cases} 1 & \rho \le 1 \\ 0 & \rho > 1 \end{cases} \qquad (13\text{-}13)$$

$$\mathcal{F}\{\mathrm{circ}(\rho)\} = \frac{J_1(2\pi f)}{f} \qquad (13\text{-}14)$$

$J_1$ is the Bessel function of the first kind[1] of order 1.




Footnote:

1. The left plot shows the various orders of the Bessel functions of the first kind. They are also called the cylindrical harmonics. Any arbitrary function in the cylindrical coordinates can be expressed as the linear superposition of these harmonics. To find the derivation of Eq (13-14), you can go to: http://mathworld.wolfram.com/CylinderFunction.html

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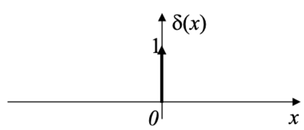
## 4.9 Delta Function

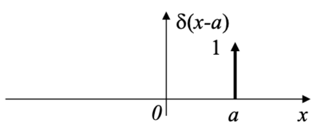
(3) Delta function



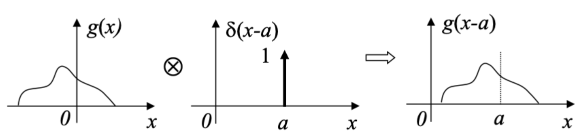

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} \delta(x) dx = 1 \qquad (13\text{-}15)$$

$$\mathcal{F}\{\delta(x)\} = 1 \qquad (13\text{-}16)$$

*Delta function with a shift:*




$$\mathcal{F}\{\delta(x-a)\} = e^{-i2\pi fa} \qquad (13\text{-}17)$$

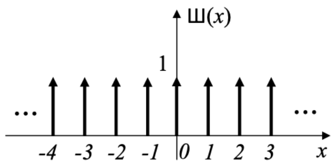



$$\mathcal{F}\{g(x) \otimes \delta(x-a)\} = e^{-i2\pi fa} \mathcal{F}\{g(x)\} \qquad (13\text{-}18)$$

## 4.10 Impulse Train

Also called Shah function

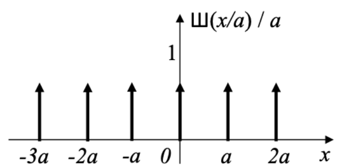
(4) Impulse Train Function (Shah function)




$$Ш(x) \equiv \sum_{n=-\infty}^{+\infty} \delta(x-n) \qquad (13\text{-}19)$$

$$\mathcal{F}\{Ш(x)\} = Ш(f) \qquad (13\text{-}20)$$

*Shah function with scaling:*




$$\frac{Ш(x/a)}{a} \equiv \sum_{n=-\infty}^{+\infty} \delta(x-na) \qquad (13\text{-}21)$$

$$\mathcal{F}\left\{\frac{Ш(x/a)}{a}\right\} = Ш(af) \qquad (13\text{-}22)$$

## 4.11 Diffraction Limit

Focal point cannot physically be a singularity. Finite wavelength size. Focal spot size $2w_o$:

$$2w_o \approx \frac{1.22\lambda f}{D}$$

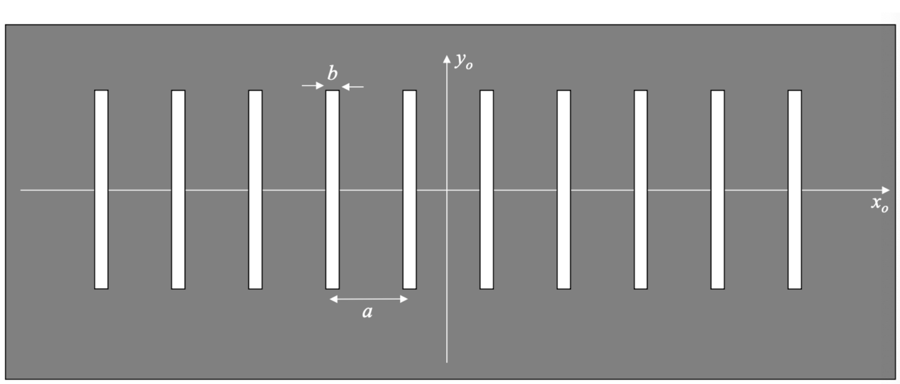
### 4.11.1 Angular Resolution

Resolving power of:

$$\theta_{RP} \approx \frac{1.22\lambda}{D}$$

## 4.12 Multi-Slit Diffraction

$N$ slits, width $b$ separation $a$



## 4.13 Diffraction Grating

$$a(\sin\theta_m - \sin\theta_i) = m\lambda$$

### 4.13.1 Resolving Power

$$\mathbb{R} = \frac{\lambda_0}{\Delta\lambda_{RP}} = mN$$

### 4.13.2 Free Spectral Range

if $\lambda_1 \approx \lambda_2$:

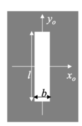
$$\Delta\lambda_{FSR} = \frac{\lambda}{m}$$

else:

$$\Delta\lambda_{FSR} = \frac{\lambda_2}{m}$$

# 5 Single-Slit Far Field Diffraction

**Example 13-1:** Find and plot the far-field diffraction pattern of a single rectangular slit of dimension $b\times l$.




Aperture function: $$E_o(x_o, y_o) = \Pi\left(\frac{x_o}{b}\right)\Pi\left(\frac{y_o}{l}\right) \quad (13\text{-}23)$$

Far-field $E$ field distribution:

$$E_i(x_i, y_i) \propto \mathcal{F}\{E_o(x_o, y_o)\}\Big|_{f_x=\frac{x_i}{\lambda z_i}, f_y=\frac{y_i}{\lambda z_i}} \quad (13\text{-}24)$$

$$\mathcal{F}\{E_o(x_o, y_o)\} = \mathcal{F}\left\{\Pi\left(\frac{x_o}{b}\right)\Pi\left(\frac{y_o}{l}\right)\right\} = bl\,\mathrm{sinc}(bf_x)\,\mathrm{sinc}(lf_y) \quad (13\text{-}25)$$

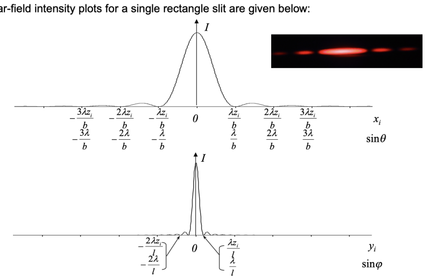
Far-field Intensity distribution:

$$I_i(x_i, y_i) \propto E_i^2(x_i, y_i) \propto \mathrm{sinc}^2(bf_x)\,\mathrm{sinc}^2(lf_y)\Big|_{f_x=\frac{x_i}{\lambda z_i}, f_y=\frac{y_i}{\lambda z_i}} = \mathrm{sinc}^2\left(\frac{bx_i}{\lambda z_i}\right)\mathrm{sinc}^2\left(\frac{ly_i}{\lambda z_i}\right) \quad (13\text{-}26)$$

One can also write the above expression in terms of $\theta$ and $\varphi$, using (13-9):

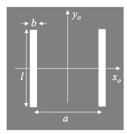
$$I_i(\theta, \varphi) \propto \mathrm{sinc}^2(bf_x)\,\mathrm{sinc}^2(lf_y)\Big|_{f_x=\frac{k\sin\theta}{2\pi}, f_y=\frac{k\sin\varphi}{2\pi}} = \mathrm{sinc}^2\left(\frac{kb\sin\theta}{2\pi}\right)\mathrm{sinc}^2\left(\frac{kl\sin\varphi}{2\pi}\right) \quad (13\text{-}27)$$

The far-field intensity plots for a single rectangle slit are given below:



# 6 Double-Slit Far Field Diffraction

**Example 13-2:** Find and plot the far-field diffraction pattern of a double slit aperture as shown.




Aperture function:

$$E_o(x_o, y_o) = \Pi\left(\frac{x_o}{b}\right)\Pi\left(\frac{y_o}{l}\right)\otimes\left[\delta\left(x+\frac{a}{2}\right)+\delta\left(x-\frac{a}{2}\right)\right] \quad (13\text{-}28)$$

Far-field $E$ field distribution:

$$E_i(x_i, y_i) \propto \mathcal{F}\{E_o(x_o, y_o)\}\Big|_{f_x=\frac{x_i}{\lambda z_i}, f_y=\frac{y_i}{\lambda z_i}} \quad (13\text{-}29)$$

$$\mathcal{F}\{E_o(x_o, y_o)\} = \mathcal{F}\left\{\Pi\left(\frac{x_o}{b}\right)\Pi\left(\frac{y_o}{l}\right)\otimes\left[\delta\left(x+\frac{a}{2}\right)+\delta\left(x-\frac{a}{2}\right)\right]\right\}$$

$$= bl\,\text{sinc}(bf_x)\,\text{sinc}(lf_y)\left(e^{i\pi f_x a}+e^{-i\pi f_x a}\right) = 2bl\cos(\pi f_x a)\,\text{sinc}(bf_x)\,\text{sinc}(lf_y) \quad (13\text{-}30)$$

Far-field Intensity distribution:

$$I_i(x_i, y_i) \propto E_i^2(x_i, y_i) \propto \cos^2(\pi f_x a)\,\text{sinc}^2(bf_x)\,\text{sinc}^2(lf_y)\Big|_{f_x=\frac{x_i}{\lambda z_i}, f_y=\frac{y_i}{\lambda z_i}}$$

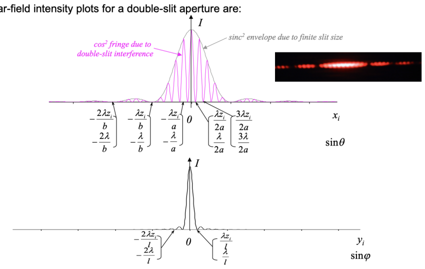
$$= \cos^2\left(\frac{\pi a x_i}{\lambda z_i}\right)\text{sinc}^2\left(\frac{b x_i}{\lambda z_i}\right)\text{sinc}^2\left(\frac{l y_i}{\lambda z_i}\right) \quad (13\text{-}31)$$

Written in terms of θ and φ:

$$I_i(\theta, \varphi) \propto \cos^2(\pi f_x a)\,\text{sinc}^2(bf_x)\,\text{sinc}^2(lf_y)\Big|_{f_x=\frac{k\sin\theta}{2\pi}, f_y=\frac{k\sin\varphi}{2\pi}}$$

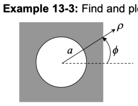
$$= \cos^2\left(\frac{ka\sin\theta}{2}\right)\text{sinc}^2\left(\frac{kb\sin\theta}{2\pi}\right)\text{sinc}^2\left(\frac{kl\sin\varphi}{2\pi}\right) \quad (13\text{-}32)$$

The far-field intensity plots for a double-slit aperture are:




# 7 Circular Aperture Diffraction

**Example 13-3:** Find and plot the far-field diffraction pattern of a circular aperture as shown.




For an aperture of radius $a$, the aperture function is expressed as:

$$E_o(\rho_o) = \text{circ}\left(\frac{\rho_o}{a}\right) \quad (13\text{-}33)$$

From (13-14), the far-field becomes:

$$E_i(\rho_i) \propto \mathcal{F}\{E_o(\rho_o)\} = \frac{1}{f_\rho}J_1(2\pi f_\rho a) \quad (13\text{-}34)$$

$$\text{where} \quad f_\rho = \frac{\rho_i}{\lambda z_i} = \frac{k\sin\theta}{2\pi} \quad (13\text{-}35)$$

Sometimes $E_i$ is expressed in terms of the angular distance $\theta$,

$$E_i(\theta) \propto \frac{2\pi}{k\sin\theta}J_1(ka\sin\theta) \quad (13\text{-}36)$$

The far-field intensity becomes:

$$I_i(\theta) \propto E_i^2(\theta) \propto \left(\frac{2\pi}{k\sin\theta}\right)^2 J_1^2(ka\sin\theta) = 4\pi^2 a^2\left(\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right)^2 \quad (13\text{-}37)$$

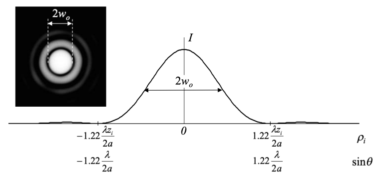
The first null of $I_i$ in Eq (13-37) occurs when

$$ka\sin\theta = 3.83 \quad \Rightarrow \quad \sin\theta = 1.22\lambda/2a \quad (13\text{-}38)$$

$$\text{or} \quad 2\pi\frac{\rho_i}{\lambda z_i}a = 3.83 \quad \text{or} \quad \rho_i = 1.22\lambda z_i/2a \quad (13\text{-}39)$$

# 8 Airy Disk

The normalized far-field intensity is plotted here:




This intensity pattern is also known as the Airy Disk.