1 Difference Equations

$$y(k)+a_1y(k-1)+\ldots+a_ny(k-n)= b_0u(k)+b_1u(k-1)+\ldots+b_mu(k-m) \quad (1)$$

1.1 Solution to Difference Equations

$$y(k)=y_h(k)+y_p(k)$$

$$\text{try} \quad y_h(k)=\lambda^k \rightarrow \lambda^n+a_1\lambda^{n-1}+\ldots+a_n=0$$

2 Laplace Transforms

2.1 Basic Laplace Table

$\mathcal{L}\{1(t)\}=\frac{1}{s}$	$\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}}$
$\mathcal{L}\{\frac{t^k}{k!}e^{at}\}=\frac{1}{(s-a)^{k+1}}$	$\mathcal{L}\{e^{at}\}=\frac{1}{s-a}$
$\mathcal{L}\{\sin(kt)\}=\frac{k}{s^2+k^2}$	$\mathcal{L}\{\cos(kt)\}=\frac{s}{s^2+k^2}$
$\mathcal{L}\{\sinh(kt)\}=\frac{k}{s^2-k^2}$	$\mathcal{L}\{\cosh(kt)\}=\frac{s}{s^2-k^2}$

2.2 Basic Inverse Laplace Table

$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1$	$\mathcal{L}^{-1}\{1\}=\delta(t)$
$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$	$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}=t^n$
$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\sin(kt)$	$\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}=\cos(kt)$
$\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}=\sinh(kt)$	$\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}=\cosh(kt)$

2.3 Forward Laplace Transform

$$\mathcal{L}\{f(t)\}=F(s):=\int_0^{+\infty}f(t)e^{-st}dt$$

2.4 Inverse Laplace Transform

$$f(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)e^{st}dt=\sum_{k=1}^{n}\mathrm{Res}(e^{st}F(s),s_k)$$

3 Z-Transforms

3.1 Forward Z-Transform

$$\mathcal{Z}\{x(k)\}=X(k):=\sum_{k=0}^{\infty}x(k)z^{-k}, \quad z\in\mathbb{C}$$

3.2 Inverse Z-Transform

$$x(k)=\mathcal{Z}^{-1}\{X(z)\}=\sum\mathrm{Res}(X(z)z^{k-1},p_z)$$

3.2.1 Simple Pole

$$\mathrm{Res}(f(z),z_0)=\lim_{z\to z_0}(z-z_0)f(z)$$

3.2.2 Pole of Order N

$$\mathrm{Res}(f(z),z_0)=\frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^nf(z)\right]\Big|_{z=z_o}$$

3.3 Linearity

For $d_1,d_2\in\mathbb{R}$

$$\mathcal{Z}\{d_1x_1(k)+d_2x_2(k)\}=d_1\mathcal{Z}\{x_1(k)\}+d_2\mathcal{Z}\{x_2(k)\}$$

3.4 Important Z-Transforms

$\mathcal{Z}\{\delta(k)\}=1$	$\mathcal{Z}\{1\}=\frac{z}{z-1}$
$\mathcal{Z}\{k\}=\frac{z}{(z-1)^2}$	$\mathcal{Z}\{k^2\}=\frac{z(z+1)}{(z-1)^3}$
$\mathcal{Z}\{a^k\}=\frac{z}{z-a}$	$\mathcal{Z}\{ka^{k-1}\}=\frac{z}{(z-a)^z}$
$\mathcal{Z}\{ka^k\}=\frac{az}{(z-a)^2}$	$\mathcal{Z}\{\frac{k!}{i!(k-i)!}a^{k-i}\}=\frac{z}{(z-a)^{i+1}}$
$\mathcal{Z}\{e^{ak}\}=\frac{z}{z-e^a}$	$\mathcal{Z}\{ke^{ak}\}=\frac{ze^a}{(z-e^a)^2}$
$\mathcal{Z}\{\sin(ak)\}=\frac{z\sin(a)}{z^2-2\cos(a)z+1}$	$\mathcal{Z}\{\cos(ak)\}=\frac{z(z-\cos(a))}{z^2-2\cos(a)z+1}$
$\mathcal{Z}\{\sin(ak)b^k\}=\frac{bz\sin(a)}{z^2-2\cos(a)bz+b^2}$	$\mathcal{Z}\{\cos(ak)b^k\}=\frac{z(z-b\cos(a))}{z^2-2\cos(a)bz+b^2}$

3.5 Convolution of Signals

For $x(k),y(k)$ and $k\geq 0$,

$$x*y=\sum_{l=-\infty}^{\infty}x(l)y(k-l)=\sum_{l=0}^{k}x(l)y(k-l)$$

3.5.1 Sifting Property

$$f(t)*\delta(t-T_0)=f(t-T_0)$$

3.6 Multiplication by a^k

$$\mathcal{Z}\{a^kx(k)\}=\sum_{k=0}^{\infty}a^kx(k)z^{-k}=X\left(\frac{z}{a}\right)$$

3.7 Forward Shift

$$\mathcal{Z}\{x(k+m)\}=z^mX(z)-\sum_{l=0}^{m-1}x(l)z^mz^{-l}$$

$$\mathcal{Z}\{x(k+m)\}=z^mX(z)-[z^mx(0)+z^{m-1}x(1)+\ldots+zx(m-1)]$$

3.8 Backward Shift

$$\mathcal{Z}\{x(k-m)\}=z^{-m}X(z)+\sum_{l=0}^{m-1}x(l-m)z^{-l}$$

$$\mathcal{Z}\{x(k-m)\}=z^{-m}X(z)+x(-m)+z^{-1}x(-m+1)+\ldots+x(-1)z^{-m+1}$$

4 Final Value Theorem

If $\lim_{k\to\infty}x(k)$ exists, then

$$\lim_{k\to\infty}x(k)=\lim_{z\to 1}(z-1)\cdot X(z)$$

4.1 FVT Existence Condition

$\lim_{k\to\infty}x(k)$ exists (finite) iff $X(z)$ has no poles in $|z|\geq 1\in\mathbb{C}$ and at most 1 pole at $z=1$

5 Initial Value Theorem

$$\lim_{k\to 0}x(k)=\lim_{z\to\infty}X(z)$$

6 Discrete Time System Models

6.1 Difference Equations

$$y(k)+a_1y(k-1)+\ldots+a_ny(k-n)=b_0u(k)+\ldots+b_mu(k-m)$$

6.2 G4 Transfer Functions

$$E(z)=\frac{1}{1+CG}R(z)+\frac{-G}{1+CG}D(z)$$

$$U(z)=\frac{C}{1+CG}R(z)+\frac{1}{1+CG}D(z)$$

7 Model Conversion

7.1 CT SS to TF

$$Y(s)=[C(sI-A)^{-1}B+D]U(s)$$

7.2 CT TF to SS

$$V(s)=\frac{1}{s^n+\cdots+a_0}U(s),\ Y(s)=(b_ms^m+\cdots+b_0)V(s)$$

$$f(x,u)=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ & &\ddots& & \\0&0&0&\cdots&1\\-a_0&-a_1&-a_2&\cdots&-a_{n-1}\end{bmatrix}x+\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}0&\cdots&0&b_0&b_1&\cdots&b_m\end{bmatrix}x$$

7.3 DT SS to TF

$$Y(z)=[C(zI-A)^{-1}B+D]U(z)$$

7.4 DT TF to SS

$$V(z)=\frac{1}{z^n+\ldots+a_0}U(z),\ Y(z)=(b_mz^m+\ldots+b_1)V(z)$$

$$A=\begin{bmatrix}1&0&\ldots&0\\ &\vdots& &\vdots\\0&0& &1\\-a_0&-a_1&\cdots&-a_{n-1}\end{bmatrix} \quad B=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$$

$$C=\begin{bmatrix}0&\ldots&0&b_1&\ldots&b_m\end{bmatrix}$$

8 Solution to State Space Models

$$x(k)=A^kx(0)+\sum_{i=0}^{k-1}A^{k-1-i}Bu(i)$$

$$y(k)=CA^k(0)+\sum_{i=0}^{k-1}CA^{k-1-i}Bu(i)$$

9 Solution to CT State Space Models

$$x(t)=e^{At}x(0)+\int_0^te^{A(t-\tau)}Bu(\tau)dt$$

9.1 Eigenvector Method for finding A^k

$$\lambda\rightarrow\det(sI-A)=0$$

$$AP=P\Lambda,\quad A^K=P\Lambda^KP^{-1},\quad Av=\lambda v$$

$$\Lambda=\begin{bmatrix}\lambda_1&\ldots&0\\\vdots&\ddots&\vdots\\0&\ldots&\lambda_n\end{bmatrix}\quad P=\begin{bmatrix}v_1&\ldots&v_n\end{bmatrix}$$

9.2 Z-Transform Method for finding A^k

Faster for $n\leq 3$

$$A^k=\mathcal{Z}^{-1}\{z(zI-A)^{-1}\}$$

10 Transient Response

10.1 Real Poles

$$\mathcal{Z}^{-1}\left[\frac{z}{z-p}\right]=p^k, k\geq 0$$

If $|p|>1$, $y(k)\to\infty$. If $p<0$, $y(k)$ alternates between +ve,-ve values. If $|p|<1$, $y(k)\to 0$

11 Sampled Data Systems

Sample, Hold operators are linear. $H\circ S$ is NOT time-invariant. $S\circ H$ is time-invariant.

11.1 Sample Operator

$$y(t)\to y_d(k)=y(kT)$$

11.2 Hold Operator

$$u_d(k)=u(kT)\to u(t)=u(kT),\ kT\ \leq t<(k+1)T$$

11.3 Discretized Plant LTI Model (HoS)

$$x((k+1)T=e^{AT}x(kT)+\int_0^Te^{As}ds\cdot Bu(kT)$$

$$A_d:=e^{AT}\qquad B_d:=\int_0^Te^{A\tau}d\tau\cdot B$$

$$G_d(z)=C_d(zI-A_d)^{-1}B_d+D_d$$

11.4 Eigenvector Method for Finding e^{AT}

$$AP=P\Lambda,\quad A=P\Lambda P^{-1},\quad Av=\lambda v$$

$$e^{At}=Pe^{\Lambda t}P^{-1}=P\begin{bmatrix}e^{\lambda_1t}& & \\ &\ldots& \\ & &e^{\lambda_nt}\end{bmatrix}P^{-1}$$

11.5 Inverse Laplace Transform for e^{AT}

$$e^{At}=\mathcal{L}^{-1}\{(sI-A)^{-1}\}$$

11.5.1 Matrix Exponential

$$e^{At}:=I+At+\frac{A^2t^2}{2!}+\ldots=\sum_{n=0}^{\infty}\frac{t^nA^n}{n!}$$

11.6 CT to DT Direct

$$G_d(z)=\frac{z-1}{z}\mathcal{Z}\left\{S\left(\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right)\right\}$$

12 Spectral Mapping Theorem

Let $A\in\mathbb{R}^{n\times n}$ and let $f:\mathbb{C}\to\mathbb{C}$ be an analytic fn at the eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$ of A. Then $f(A)$ is a matrix with eigenvalues $\{f(\lambda_1),\ldots,f(\lambda_n)\}$.

$$\frac{N(s)}{(s-p_1)\ldots(s-p_n)}\to\frac{N_d(z)}{(z-e^{p_1t})\ldots(z-e^{p_nt})}$$

13 Fourier Transforms

$$y(j\omega)=\mathcal{F}\{y(t)\}=\int_{-\infty}^{\infty}y(\tau)e^{-j\omega\tau}d\tau$$

$$y_d(e^{j\omega t})=\mathcal{F}\{y_d(k)\}=\sum_{k=0}^{\infty}y_d(k)e^{-j\omega Tk}$$

13.1 Convolutions of Fourier Transforms

$$\mathcal{F}\{x_1(t)\cdot x_2(t)\}=\frac{1}{2\pi}X_1(j\omega)*X_2(j\omega)$$

14 Periodic Extension

$$y_e(j\omega)=\sum_{k=-\infty}^{\infty}y(j\omega+jk\frac{2\pi}{T})$$

15 CT Frequency Response

$$G(s)|_{s=j\omega}=G(j\omega),\ \omega\in[0,\infty)$$

$$y(t)=G(j\omega)e^{j\omega t}$$

16 DT Frequency Response

$$G_d(z)|_{z=e^{j\omega T}}=G_d(e^{j\omega T})$$

$$y(k)=\left[G_d(e^{j\omega T})\right]\cdot e^{j\omega Tk}$$

16.1 DC Gain

$$G(s)|_{s=0}=G_d(z)|_{z=1}$$

16.2 DC Freq. Response and Artifact of Sampling

$$G_d(e^{j\theta})=G_d(e^{j(\theta+2\pi)})$$

17 Sample Operator in Frequency

$$V(j\omega)=\mathcal{F}\left\{y(t)\cdot\sum_{k=-\infty}^{\infty}\delta(t-KT)\right\}$$

$$V(j\omega)=\frac{1}{2\pi}y(j\omega)*\mathcal{L}\left\{\sum_{k=-\infty}^{\infty}\delta(t-kT)\right\}$$

$$y_d(e^{j\omega T})=\frac{1}{T}y_e(j\omega)$$

17.1 Fourier Transform of Impulse Train

$$\mathcal{L}\left\{\sum_{k=-\infty}^{\infty}\delta(t-kT)\right\}=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\frac{2\pi}{T}k)$$

18 Hold Operator in Frequency

$$r(t)=\frac{1}{T}[1(t)-1(t-T)]\to R(s)=\frac{1}{T}\left[\frac{1}{s}-\frac{e^{-Ts}}{s}\right]$$

$$U(j\omega) = TR(j\omega)U_d(e^{j\omega T})$$

19 Discrete Time and Frequency Domain

For a $H \circ G(s) \circ S$ system ($u_d(k) \to G_d(z) \to y_d(k)$)

$$G_d(e^{j\omega T}) = \frac{Y_d(e^{j\omega T})}{U_d(e^{j\omega T})} \qquad R(j\omega) = \frac{1-e^{-j\omega T}}{j\omega T}$$

$$G_d(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} G(j\omega + j\frac{2\pi}{T}k) \cdot R(j\omega + j\frac{2\pi}{T}k)$$

20 Nyquist-Shannon Sampling Theorem

If $G_d(e^{j\omega t})$ known, and $|G(j\omega)| = 0$ for $|\omega| \geq \frac{\pi}{T}$ (Nyquist Frequency), then $G(j\omega)$ is recoverable ($G_d(z) \to G(s)$).

$$G_d(e^{j\omega T}) \approx G(j\omega) \qquad |\omega| << \frac{\pi}{T}$$

21 Stability

An DT system $x(k+1) = Ax(k)$ is asy. stable if

$$x(k) = A^k x(0) \to 0, \quad \forall x(0)$$
$$A^k \to \text{as } k \to \infty$$

A DT system is stable if

$$\forall x(0),\ x(k) \leq M \quad \forall k \geq 0$$

21.1 Asymptotic Stability

A system is AS iff $|\lambda| < 1 \ \forall\ \lambda \in \sigma(A)$

21.2 Internal Stability

A system is stable iff $|\lambda| \leq 1 \forall \lambda \in \sigma(A)$. Additionally, for any $\lambda \in \sigma(A)$ with $|\lambda| = 1$ and λ has multiplicity $k \geq 1$, there must be k linearly independent eigenvectors (A_i is diagonalizable).

22 Controlability

22.1 Controlability Matrix

$$Q_c = [B \quad AB \quad \ldots \quad A^{n-1}B] \qquad Q_c \in \mathbb{R}^{n \times n \cdot m}$$

A pair (A,B) is controllable if $\text{rank}(Q_c) = n$

22.2 PBH Test

(A,B) is **controllable** iff for eigenvalues $\lambda \in \sigma(A)$

$$\text{rank}[A - \lambda I \quad B] = n$$

(A,B) is **stabilizable** if $\exists F$ s.t. $\sigma(A+BF) \in \{|z| < 1\}$

$$(rank)[A - \lambda I \quad B] = n$$

for each eigenvalue $\lambda \in \sigma(A)$ with $|\lambda| \geq 1$

22.3 Controlable Canonical Form (CCF)

$$x(k+1) = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & 1 \\ a_1 & a_2 & \ldots & a_n \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

22.4 Cayleigh-Hamilton Theorem

Every square matrix A satisfies its own characteristic polynomial

$$\Delta(A) = 0 \implies A^n = -a_1 A^{n-1} - \ldots - a_n I$$

23 Pole Placement Theorem

Given $p_1, \ldots, p_n$ desired CLS poles, and using state feedback $u(k) = [F_1 \quad \ldots \quad F_n]x(k)$

1. $r(z) = (z-p_1)(z-p_2)\ldots(z-p_n)$
2. Convert (A,B) to CCF $(\overline{A}, \overline{B})$
3. Compute $\Delta(z) = \det(zI - (\overline{A} + \overline{BF}))$
4. Match coefficients $\Delta(z) = \Delta_d(z) = r(z)$
5. $F = \overline{F}P^{-1}$

23.1 P, P^{-1} and Related Matrices

$$Q_c = W = [B \quad AB \quad \ldots \quad A^{n-1}B]$$
$$\overline{Q_c} = [\overline{B} \quad \overline{AB} \quad \ldots \quad \overline{A}^{n-1}\overline{B}]$$
$$P = Q_c \overline{Q_c}^{-1}$$

23.2 Deadbeat Control

If (A,B) controllable, assign all

$$\sigma(A+BK) = \{0, \ldots, 0\} \implies \Delta s = s^n$$

24 Ackermann's Formula

Let $\{\lambda_{1d}, \ldots, \lambda_{nd}\}$ be the desired poles of $A + BK$

$$\Delta_d(z) = (z - \lambda_{1d})\ldots(z - \lambda_{nd}) = z^n + r_1 z^{n-1} + \ldots + r_n$$
$$K = -[0 \quad \ldots \quad 0 \quad 1]Q_c^{-1}\Delta_d(A)$$
$$\Delta_d(A) = A^n + r_1 A^{n-1} + \ldots + r_n I$$

24.1 Stabilizability

A system is stabilizable if

$$\exists K \in \mathbb{R}^{n \times m} \text{ s.t. } \sigma(A_d + B_d K) \subset \{z \in \mathbb{C}, |z| < 1\}$$

25 Observability

25.1 Observability Matrix

$$Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad Q_o \in \mathbb{R}^{n \cdot (p \times n)}$$

A pair (C,A) is observable if $\text{rank}(Q_o) = n$ (full col rank).

25.2 PBH Test

(C,A) is **observable** iff for eigenvalues $\lambda \in \sigma(A)$

$$\text{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$$

(C,A) is **detectable** if $\exists L$ s.t. $\sigma(A - LC) \in \{|z| < 1\}$

$$\text{rank}\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$$

for each eigenvalue $\lambda \in \sigma(A)$ with $|\lambda| \geq 1$

25.3 Observers and Dynamic Compensation

Assuming (A,B) controllable, (C,A) observable.

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k))$$
$$\hat{y}(k) = C\hat{x}(k)$$

25.4 Estimation Error

$$\tilde{x}(k) = x(k) - \hat{x}(k) \qquad \tilde{x}(k+1) = (A - LC)\tilde{x}(k)$$

25.5 Observer Based Control

$$u(k) = K\hat{x}(k)$$
$$\begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} (A+BK) & -BK \\ 0 & (A-LC) \end{bmatrix} \cdot \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$$

25.6 Separation Principle

$$\sigma(A_{cl}) = \sigma(A+BK) \cup \sigma(A-LC)$$

26 Minimal Order Observers

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad y = [I \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\hat{x}(k) = \begin{bmatrix} y(k) \\ v(k) + Ly(k) \end{bmatrix} \qquad v(k) = \hat{x_2}(k) - Lx_1(k)$$

$$v(k+1) = (A_{22} - LA_{12})v(k) + (B_2 - LB_1)u(k)$$
$$+ (A_{21} - LA_{11})x_1(k) - (A_{22} - LA_{12})Lx_1(k)$$

27 Duality Theory

$$(C,A) \text{ observable} \iff (A^T, C^T) \text{ controllable}$$
$$(C,A) \text{ detectable} \iff (A^T, C^T) \text{ stabilizable}$$
$$\text{controllable} \implies \text{stabilizable}$$
$$\text{observable} \implies \text{detectable}$$

28 Pathological Sampling

A freq $\omega_S = \frac{2\pi}{T}$ is pathological if

$$e^{\lambda_1 T} = e^{\lambda_2 T} \qquad , \lambda_1, \lambda_2 \in \sigma(A)$$

$$\lambda_i = \lambda_j + j \cdot \frac{2\pi}{T} l \qquad i, j \in \{i, \ldots, n\},\ l \in \mathbb{Z}$$

29 Exosystem

$$\dot{\omega} = S\omega \qquad \omega(k+1) = S\omega(k) \qquad r = C_2\omega$$

29.1 Common Exosystems

$r(t)/r(k)$	S	ω
$\sin(at)$	$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$	$\begin{bmatrix} r \\ \dot{r} \end{bmatrix}$
$1(kT)$	1	$r(k)$
$kT \cdot 1(kT)$	$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} r(k) \\ r(k+1) \end{bmatrix}$
$\sin(akT) \cdot 1(kT)$	$\begin{bmatrix} \cos(aT) & \sin(aT) \\ -\sin(aT) & \cos(aT) \end{bmatrix}$	$\begin{bmatrix} r(k) \\ r(k+1) \end{bmatrix}$

30 Regulator Problem

$$x(k+1) = Ax(k) + Bu(k) + E\omega(k)$$
$$\omega(k+1) = S\omega(k) \quad e(k) = Cx(k) + D\omega(k)$$

30.1 Regulator Equations

$$\pi S = A\pi + B\Gamma + E \quad 0 = c\pi + D$$

30.2 Regulator Model

$$z(k) = x(k) - \pi\omega(k)$$
$$z(k+1) = Az(k) + B(u(k) - \Gamma\omega(k)) \quad e(k) = Cz(k)$$

30.3 State Feedback (Full State Measurement)

$$u(k) = K(x(k) - \pi\omega(k)) + \Gamma\omega(k)$$

30.4 State Feedback + Observers

Assuming (A,B) controllable, (C,A) observable. Assuming $\left([C \quad D], \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}\right)$ observable

30.4.1 Estimation Error

$$\tilde{x}(k) = x(k) - \hat{x}(k) \quad \tilde{\omega}(k) = \omega(k) - \hat{\omega}(k)$$

30.4.2 Observer Based Control

$$u(k) = \Gamma\hat{\omega}(k) + K(\hat{x}(k) - \pi\hat{\omega}(k))$$
$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + E\hat{\omega}(k) + L_1(e(k) - \hat{e}(k))$$
$$\hat{\omega}(k+1) = S\hat{\omega}(k) + L_2(e(k) - \hat{e}(k))$$
$$\hat{e}(k) = C\hat{x}(k) + D\hat{\omega}(k)$$

$$\overline{A} = \begin{bmatrix} A_d & E \\ 0 & S \end{bmatrix}, \overline{B} = \begin{bmatrix} B_d \\ 0 \end{bmatrix}, \overline{C} = [C_d \quad D_d]$$

$$\begin{bmatrix} \hat{x}' \\ \hat{\omega}' \end{bmatrix} = \overline{A} \cdot \begin{bmatrix} \hat{x} \\ \hat{\omega} \end{bmatrix} + \overline{B}u(k) + \overline{L}\left(e(k) - \overline{C}\begin{bmatrix} \hat{x} \\ \hat{\omega} \end{bmatrix}\right)$$

30.5 Regulator Design

1. Design K using pole placement
2. Solve regulator problem equations for (π, Γ)
3. Write state feedback solution (Sec. 30.3)
4. Design Observers (Sec. 30.4.2)
5. Write observer-based controller s.t. $\sigma(\overline{A} - \overline{L} \cdot \overline{C}) \in \mathbb{C}^2)$ (Sec. 30.4.2)

31 Discretization of CT Controllers

31.1 c2d (Step Invariance) Method

$$C_d(z) = \text{c2d}(C(s)) = \frac{z-1}{z}\mathscr{Z}\left\{S\left(\mathscr{L}^{-1}\left\{\frac{C(s)}{s}\right\}\right)\right\}$$

31.2 Bilinear Transformation

$$s = \frac{2}{T}\frac{z-1}{z+1} \qquad C_d(z) = C\left(\frac{2}{T}\frac{z-1}{z+1}\right)$$

31.3 Pole-Zero Matching

For $C(s)$ with $d = \#\text{ poles} - \#\text{ zeros} \geq 1$, there are d infinite zeros.

$$C(s) = K\frac{(s+b_1)(s+b_2)\ldots(s+b_m)}{(s+a_1)(s+a_2)\ldots(s+a_n)} \qquad n \geq m$$

$$C_d(z) = k_d\frac{(z+1)^d(z - e^{-b_1 T})\ldots(z - e^{-b_m T})}{(z - e^{-a_1 T})\ldots(z - e^{-a_n T})}$$

Only add factor $(z+1)^d$ to num of $C_d(z)$ if $d = n - m > 0$. Choose K_d s.t. $C_d(1) = C(0)$