ECE411 Course Notes

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1 Difference Equations

$$y(k) + a_1 y(k-1) + \ldots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \ldots + b_m u(k-m) \quad (1)$$

1.1 Solution to Difference Equations

$$y(k) = y_h(k) + y_p(k)$$
$$\text{try} \quad y_h(k) = \lambda^k \rightarrow \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n = 0$$

2 Laplace Transforms

2.1 Basic Laplace Table

X — X	$\mathscr{L}\{1(t)\} = \frac{1}{s}$	$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$
	$\mathscr{L}\{\frac{t^k}{k!} e^{at}\} = \frac{1}{(s-a)^{k+1}}$	$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$
	$\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$	$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$
	$\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$	$\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$

2.2 Basic Inverse Laplace Table

X — X	$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$	$\mathscr{L}^{-1}\{1\} = \delta(t)$
	$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$	$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$
	$\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$	$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$
	$\mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh(kt)$	$\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt)$

2.3 Forward Laplace Transform

$$\mathscr{L}\{f(t)\} = F(s) \coloneqq \int_0^{+\infty} f(t) e^{-st} dt$$

2.4 Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) e^{st} dt = \sum_{k=1}^{n} \mathrm{Res}(e^{st} F(s), s_k)$$

3 Z-Transforms

3.1 Forward Z-Transform

$$\mathscr{Z}\{x(k)\} = X(k) \coloneqq \sum_{k=0}^{\infty} x(k) z^{-k}, \quad z \in \mathbb{C}$$

3.2 Inverse Z-Transform

$$x(k) = \mathscr{Z}^{-1}\{X(z)\} = \sum \mathrm{Res}(X(z) z^{k-1}, p_z)$$

3.2.1 Simple Pole

$$\mathrm{Res}(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

3.2.2 Pole of Order N

$$\mathrm{Res}(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z)\right] \bigg|_{z = z_o}$$

3.3 Linearity

For $d_1, d_2 \in \mathbb{R}$

$$\mathscr{Z}\{d_1 x_1(k) + d_2 x_2(k)\} = d_1 \mathscr{Z}\{x_1(k)\} + d_2 \mathscr{Z}\{x_2(k)\}$$

3.4 Important Z-Transforms

X — X $\mathscr{Z}\{\delta(k)\} = 1$ $\mathscr{Z}\{1\} = \frac{z}{z-1}$

$\mathscr{Z}\{k\} = \frac{z}{(z-1)^2}$ $\mathscr{Z}\{k^2\} = \frac{z(z+1)}{(z-1)^3}$

$\mathscr{Z}\{a^k\} = \frac{z}{z-a}$ $\mathscr{Z}\{ka^{k-1}\} = \frac{z}{(z-a)^z}$

$\mathscr{Z}\{ka^k\} = \frac{az}{(z-a)^2}$ $\mathscr{Z}\{\frac{k!}{i!(k-i)!} a^{k-i}\} = \frac{z}{(z-a)^{i+1}}$

$\mathscr{Z}\{e^{ak}\} = \frac{z}{z-e^a}$ $\mathscr{Z}\{ke^{ak}\} = \frac{ze^a}{(z-e^a)^2}$

$\mathscr{Z}\{\sin(ak)\} = \frac{z \sin(a)}{z^2 - 2\cos(a) z + 1}$ $\mathscr{Z}\{\cos(ak)\} = \frac{z(z - \cos(a))}{z^2 - 2\cos(a) z + 1}$

$\mathscr{Z}\{\sin(ak) b^k\} = \frac{bz \sin(a)}{z^2 - 2\cos(a) bz + b^2}$ $\mathscr{Z}\{\cos(ak) b^k\} = \frac{z(z - b\cos(a))}{z^2 - 2\cos(a) bz + b^2}$

3.5 Convolution of Signals

For $x(k), y(k)$ and $k \geq 0$,

$$x * y = \sum_{l=-\infty}^{\infty} x(l)y(k-l) = \sum_{l=0}^{k} x(l)y(k-l)$$

3.5.1 Sifting Property

$$f(t) * \delta(t - T_0) = f(t - T_0)$$

3.6 Multiplication by a^k

$$\mathscr{Z}\{a^k x(k)\} = \sum_{k=0}^{\infty} a^k x(k) z^{-k} = X\left(\frac{z}{a}\right)$$

3.7 Forward Shift

$$\mathscr{Z}\{x(k+m)\} = z^m X(z) - \sum_{l=0}^{m-1} x(l) z^m z^{-l}$$
$$\mathscr{Z}\{x(k+m)\} = z^m X(z) - [z^m x(0) + z^{m-1} x(1) + \ldots + z x(m-1)]$$

3.8 Backward Shift

$$\mathscr{Z}\{x(k-m)\} = z^{-m} X(z) + \sum_{l=0}^{m-1} x(l-m) z^{-l}$$
$$\mathscr{Z}\{x(k-m)\} = z^{-m} X(z) + x(-m) + z^{-1} x(-m+1) + \ldots + x(-1) z^{-m+1}$$

4 Final Value Theorem

If $\lim_{k\to\infty} x(k)$ exists, then

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1} (z-1) \cdot X(z)$$

4.1 FVT Existence Condition

$\lim_{k\to\infty} x(k)$ exists (finite) iff $X(z)$ has no poles in $|z| \geq 1 \in \mathbb{C}$ and at most 1 pole at $z = 1$

5 Initial Value Theorem

$$\lim_{k\to 0} x(k) = \lim_{z\to\infty} X(z)$$

6 Discrete Time System Models

6.1 Difference Equations

$$y(k) + a_1 y(k-1) + \ldots + a_n y(k-n) = b_0 u(k) + \ldots + b_m u(k-m)$$

6.2 G4 Transfer Functions

$$E(z) = \frac{1}{1+CG} R(z) + \frac{-G}{1+CG} D(z)$$
$$U(z) = \frac{C}{1+CG} R(z) + \frac{1}{1+CG} D(z)$$

7 Model Conversion

7.1 CT SS to TF

$$Y(s) = [C(sI-A)^{-1}B + D]U(s)$$

7.2 CT TF to SS

$$V(s) = \frac{1}{s^n + \cdots + a_0} U(s) \text{ , } Y(s) = (b_m s^m + \cdots + b_0)V(s)$$
$$f(x,u) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_m \end{bmatrix} x$$

7.3 DT SS to TF

$$Y(z) = [C(zI-A)^{-1}B + D]U(z)$$

7.4 DT TF to SS

$$V(z) = \frac{1}{z^n + \ldots + a_0} U(z) \text{ , } Y(z) = (b_m z^m + \ldots + b_1)V(z)$$
$$A = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & \ldots & 0 & b_1 & \ldots & b_m \end{bmatrix}$$

8 Solution to State Space Models

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-1-i} B u(i)$$
$$y(k) = CA^k(0) + \sum_{i=0}^{k-1} CA^{k-1-i} B u(i)$$

9 Solution to CT State Space Models

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) dt$$

9.1 Eigenvector Method for finding A^k

$$\lambda \rightarrow \det(sI - A) = 0$$
$$AP = P\Lambda, \quad A^K = P\Lambda^K P^{-1}, \quad Av = \lambda v$$
$$\Lambda = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \quad P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$$

9.2 Z-Transform Method for finding A^k

Faster for $n \leq 3$

$$A^k = \mathscr{Z}^{-1}\{z(zI - A)^{-1}\}$$

10 Transient Response

10.1 Real Poles

$$\mathscr{Z}^{-1}\left[\frac{z}{z-p}\right] = p^k, k \geq 0$$

If $|p| > 1$, $y(k) \rightarrow \infty$. If $p < 0$, $y(k)$ alternates between +ve,-ve values. If $|p| < 1$, $y(k) \rightarrow 0$

11 Sampled Data Systems

Sample, Hold operators are linear. $H \circ S$ is NOT time-invariant. $S \circ H$ is time-invariant.

11.1 Sample Operator

$$y(t) \rightarrow y_d(k) = y(kT)$$

11.2 Hold Operator

$$u_d(k) = u(kT) \rightarrow u(t) = u(kT)\ ,\ kT\ \leq t < (k+1)T$$

11.3 Discretized Plant LTI Model (HoS)

$$x((k+1)T = e^{AT}x(kT) + \int_0^T e^{As}ds \cdot Bu(kT)$$
$$A_d \coloneqq e^{AT} \qquad B_d \coloneqq \int_0^T e^{A\tau}d\tau \cdot B$$
$$G_d(z) = C_d(zI - A_d)^{-1}B_d + D_d$$

11.4 Eigenvector Method for Finding e^{AT}

$$AP = P\Lambda, \quad A = P\Lambda P^{-1}, \quad Av = \lambda v$$
$$e^{At} = Pe^{\Lambda t}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & \\ & \dots & \\ & & e^{\lambda_n t}\end{bmatrix}P^{-1}$$

11.5 Inverse Laplace Transform for e^{AT}

$$e^{At} = \mathscr{L}^{-1}\{(sI - A)^{-1}\}$$

11.5.1 Matrix Exponential

$$e^{At} \coloneqq I + At + \frac{A^2t^2}{2!} + \ldots = \sum_{n=0}^{\infty}\frac{t^nA^n}{n!}$$

11.6 CT to DT Direct

$$G_d(z) = \frac{z-1}{z}\mathscr{Z}\left\{S\left(\mathscr{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right)\right\}$$

12 Spectral Mapping Theorem

Let $A \in \mathbb{R}^{n\times n}$ and let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an analytic fn at the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of A. Then $f(A)$ is a matrix with eigenvalues $\{f(\lambda_1), \ldots, f(\lambda_n)\}$.

$$\frac{N(s)}{(s-p_1)\ldots(s-p_n)} \rightarrow \frac{N_d(z)}{(z-e^{p_1 t})\ldots(z-e^{p_n t})}$$

13 Fourier Transforms

$$y(j\omega) = \mathbb{F}\{y(t)\} = \int_{-\infty}^{\infty} y(\tau)e^{-j\omega\tau}d\tau$$

$$y_d(e^{j\omega t}) = \mathbb{F}\{y_d(k)\} = \sum_{k=0}^{\infty} y_d(k)e^{-j\omega Tk}$$

13.1 Convolutions of Fourier Transforms

$$\mathbb{F}\{x_1(t) \cdot x_2(t)\} = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

14 Periodic Extension

$$y_e(j\omega) = \sum_{k=-\infty}^{\infty} y(j\omega + jk\frac{2\pi}{T})$$

15 CT Frequency Response

$$G(s)|_{s=j\omega} = G(j\omega)\ ,\ \omega \in [0, \infty)$$
$$y(t) = G(j\omega)e^{j\omega t}$$

16 DT Frequency Response

$$G_d(z)|_{z=e^{j\omega T}} = G_d(e^{j\omega T})$$
$$y(k) = \left[G_d(e^{j\omega T})\right] \cdot e^{j\omega Tk}$$

16.1 DC Gain

$$G(s)|_{s=0} = G_d(z)|_{z=1}$$

16.2 DC Freq. Response and Artifact of Sampling

$$G_d(e^{j\theta}) = G_d(e^{j(\theta+2\pi)})$$

17 Sample Operator in Frequency

$$V(j\omega) = \mathbb{F}\left\{y(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - KT)\right\}$$
$$V(j\omega) = \frac{1}{2\pi} y(j\omega) * \mathscr{L}\left\{\sum_{k=-\infty}^{\infty} \delta(t - kT)\right\}$$
$$y_d(e^{j\omega T}) = \frac{1}{T} y_e(j\omega)$$

17.1 Fourier Transform of Impulse Train

$$\mathscr{L}\left\{\sum_{k=-\infty}^{\infty}\delta(t-kT)\right\} = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega-\frac{2\pi}{T}k)$$

18 Hold Operator in Frequency

$$r(t) = \frac{1}{T}\left[1(t)-1(t-T)\right] \rightarrow R(s) = \frac{1}{T}\left[\frac{1}{s}-\frac{e^{-Ts}}{s}\right]$$
$$U(j\omega) = TR(j\omega)U_d(e^{j\omega T})$$

19 Discrete Time and Frequency Domain

For a $H \circ G(s) \circ S$ system $(u_d(k) \rightarrow G_d(z) \rightarrow y_d(k))$

$$G_d(e^{j\omega T}) = \frac{Y_d(e^{j\omega T})}{U_d(e^{j\omega T})} \qquad R(j\omega) = \frac{1-e^{-j\omega T}}{j\omega T}$$
$$G_d(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} G(j\omega + j\frac{2\pi}{T}k)\cdot R(j\omega + j\frac{2\pi}{T}k)$$

20 Nyquist-Shannon Sampling Theorem

If $G_d(e^{j\omega t})$ known, and $|G(j\omega)| = 0$ for $|\omega| \geq \frac{\pi}{T}$ (Nyquist Frequency), then $G(j\omega)$ is recoverable $(G_d(z) \rightarrow G(s))$.

$$G_d(e^{j\omega T}) \approx G(j\omega) \qquad |\omega| << \frac{\pi}{T}$$

21 Stability

An DT system $x(k+1) = Ax(k)$ is asy. stable if

$$x(k) = A^k x(0) \rightarrow 0, \quad \forall x(0)$$
$$A^k \rightarrow \text{ as } k \rightarrow \infty$$

A DT system is stable if

$$\forall x(0)\;,\; x(k) \leq M \quad \forall k \geq 0$$

21.1 Asymptotic Stability

A system is AS iff $|\lambda| < 1\,\forall\,\lambda \in \sigma(A)$

21.2 Internal Stability

A system is stable iff $|\lambda| \leq 1 \, \forall \, \lambda \in \sigma(A)$. Additionally, for any $\lambda \in \sigma(A)$ with $|\lambda| = 1$ and λ has multiplicity $k \geq 1$, there must be k linearly independent eigenvectors (A_i is diagonalizable).

22 Controlability

22.1 Controlability Matrix

$$Q_c = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \quad Q_c \in \mathbb{R}^{n \times n \cdot m}$$

A pair (A, B) is controllable if $\text{rank}(Q_c) = n$

22.2 PBH Test

(A, B) is **controllable** iff for eigenvalues $\lambda \in \sigma(A)$

$$\text{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$$

(A, B) is **stabilizable** if $\exists F$ s.t. $\sigma(A + BF) \in \{|z| < 1\}$

$$(rank) \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$$

for each eigenvalue $\lambda \in \sigma(A)$ with $|\lambda| \geq 1$

22.3 Controlable Canonical Form (CCF)

$$x(k+1) = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & 1 \\ a_1 & a_2 & \dots & a_n \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

22.4 Cayleigh-Hamilton Theorem

Every square matrix A satisfies its own characteristic polynomial

$$\Delta(A) = 0 \implies A^n = -a_1 A^{n-1} - \dots - a_n I$$

23 Pole Placement Theorem

Given $p_1, \dots, p_n$ desired CLS poles, and using state feedback $u(k) = \begin{bmatrix} F_1 & \dots & F_n \end{bmatrix} x(k)$

1. $r(z) = (z - p_1)(z - p_2) \dots (z - p_n)$
2. Convert (A, B) to CCF $(\bar{A}, \bar{B})$
3. Compute $\Delta(z) = \det(zI - (\bar{A} + \bar{B}\bar{F}))$
4. Match coefficients $\Delta(z) = \Delta_d(z) = r(z)$
5. $F = \bar{F}P^{-1}$

23.1 P, P^{-1} and Related Matrices

$$Q_c = W = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$
$$\bar{Q_c} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \dots & \bar{A}^{n-1}\bar{B} \end{bmatrix}$$
$$P = Q_c \bar{Q_c}^{-1}$$

23.2 Deadbeat Control

If (A, B) controllable, assign all

$$\sigma(A + BK) = \{0, \dots, 0\} \implies \Delta s = s^n$$

24 Ackermann's Formula

Let $\{\lambda_{1d}, \dots, \lambda_{nd}\}$ be the desired poles of $A + BK$

$$\Delta_d(z) = (z - \lambda_{1d}) \dots (z - \lambda_{nd}) = z^n + r_1 z^{n-1} + \dots + r_n$$
$$K = -\begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} Q_c^{-1} \Delta_d(A)$$
$$\Delta_d(A) = A^n + r_1 A^{n-1} + \dots + r_n I$$

24.1 Stabilizability

A system is stabilizable if

$$\exists K \in \mathbb{R}^{n \times m} \text{ s.t. } \sigma(A_d + B_d K) \subset \{z \in \mathbb{C}, |z| < 1\}$$

25 Observability

25.1 Observability Matrix

$$Q_o = \begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix} \quad Q_o \in \mathbb{R}^{n \cdot (p \times n)}$$

A pair (C, A) is observable if $\text{rank}(Q_o) = n$ (full col rank).

25.2 PBH Test

(C, A) is **observable** iff for eigenvalues $\lambda \in \sigma(A)$

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$$

(C, A) is **detectable** if $\exists L$ s.t. $\sigma(A - LC) \in \{|z| < 1\}$

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$$

for each eigenvalue $\lambda \in \sigma(A)$ with $|\lambda| \geq 1$

25.3 Observers and Dynamic Compensation

Assuming (A, B) controllable, (C, A) observable.

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C\hat{x}(k) \end{aligned}$$

25.4 Estimation Error

$$\tilde{x}(k) = x(k) - \hat{x}(k) \quad \tilde{x}(k+1) = (A - LC)\tilde{x}(k)$$

25.5 Observer Based Control

$$u(k) = K\hat{x}(k)$$
$$\begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} (A+BK) & -BK \\ 0 & (A-LC) \end{bmatrix} \cdot \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}$$

25.6 Separation Principle

$$\sigma(A_{cl}) = \sigma(A+BK) \cup \sigma(A-LC)$$

26 Minimal Order Observers

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\hat{x}(k) = \begin{bmatrix} y(k) \\ \nu(k) + Ly(k) \end{bmatrix} \quad \nu(k) = \hat{x_2}(k) - Lx_1(k)$$
$$\begin{aligned} \nu(k+1) = (A_{22} - LA_{12})\nu(k) + (B_2 - LB_1)u(k) \\ +(A_{21} - LA_{11})x_1(k) - (A_{22} - LA_{12})Lx_1(k) \end{aligned}$$

27 Duality Theory

$$\begin{aligned} (C, A) \text{ observable} &\iff (A^T, C^T) \text{ controllable} \\ (C, A) \text{ detectable} &\iff (A^T, C^T) \text{ stabilizable} \\ \text{controllable} &\implies \text{stabilizable} \\ \text{observable} &\implies \text{detectable} \end{aligned}$$

28 Pathological Sampling

A freq $\omega_S = \frac{2\pi}{T}$ is pathological if

$$e^{\lambda_1 T} = e^{\lambda_2 T} \quad , \lambda_1, \lambda_2 \in \sigma(A)$$

$$\lambda_i = \lambda_j + j \cdot \frac{2\pi}{T} l \quad i,j \in \{i,\dots,n\},\ l \in \mathbb{Z}$$

29 Exosystem

$$\dot{\omega} = S\omega \quad \omega(k+1) = S\omega(k) \quad r = C_2\omega$$

29.1 Common Exosystems

c — c — c

$r(t)$ / $r(k)$	S	ω
$\sin(at)$	$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$	$\begin{bmatrix} r \\ \dot{r} \end{bmatrix}$
$1(kT)$	1	$r(k)$
$kT \cdot 1(kT)$	$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$	$\begin{bmatrix} r(k) \\ r(k+1) \end{bmatrix}$
$\sin(akT) \cdot 1(kT)$	$\begin{bmatrix} \cos(aT) & \sin(aT) \\ -\sin(aT) & \cos(aT) \end{bmatrix}$	$\begin{bmatrix} r(k) \\ r(k+1) \end{bmatrix}$

30 Regulator Problem

$$x(k+1) = Ax(k) + Bu(k) + E\omega(k)$$
$$\omega(k+1) = S\omega(k) \quad e(k) = Cx(k) + D\omega(k)$$

30.1 Regulator Equations

$$\pi S = A\pi + B\Gamma + E \quad 0 = c\pi + D$$

30.2 Regulator Model

$$z(k) = x(k) - \pi\omega(k)$$
$$z(k+1) = Az(k) + B(u(k) - \Gamma\omega(k)) \quad e(k) = Cz(k)$$

30.3 State Feedback (Full State Measurement)

$$u(k) = K(x(k) - \pi\omega(k)) + \Gamma\omega(k)$$

30.4 State Feedback + Observers

Assuming (A, B) controllable, (C, A) observable. Assuming $\left(\begin{bmatrix} C & D \end{bmatrix}, \begin{bmatrix} A & E \\ 0 & S \end{bmatrix}\right)$ observable

30.4.1 Estimation Error

$$\tilde{x}(k) = x(k) - \hat{x}(k) \quad \tilde{\omega}(k) = \omega(k) - \hat{\omega}(k)$$

30.4.2 Observer Based Control

$$\begin{gathered}
u(k) = \Gamma\hat{\omega}(k) + K(\hat{x}(k) - \pi\hat{\omega}(k)) \\
\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + E\hat{\omega}(k) + L_1(e(k) - \hat{e}(k)) \\
\hat{\omega}(k+1) = S\hat{\omega}(k) + L_2(e(k) - \hat{e}(k)) \\
\hat{e}(k) = C\hat{x}(k) + D\hat{\omega}(k)
\end{gathered}$$

$$\bar{A} = \begin{bmatrix} A_d & E \\ 0 & S \end{bmatrix}, \bar{B} = \begin{bmatrix} B_d \\ 0 \end{bmatrix}, \bar{C} = \begin{bmatrix} C_d & D_d \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}' \\ \hat{\omega}' \end{bmatrix} = \bar{A} \cdot \begin{bmatrix} \hat{x} \\ \hat{\omega} \end{bmatrix} + \bar{B}u(k) + \bar{L}\left(e(k) - \bar{C}\begin{bmatrix} \hat{x} \\ \hat{\omega} \end{bmatrix} \right)$$

30.5 Regulator Design

1. Design K using pole placement
2. Solve regulator problem equations for (π, Γ)
3. Write state feedback solution (Sec. 30.3)
4. Design Observers (Sec. 30.4.2)
5. Write observer-based controller s.t. $\sigma(\bar{A} - \bar{L} \cdot \bar{C}) \in \mathbb{C}^2)$ (Sec. 30.4.2)

31 Discretization of CT Controllers

31.1 c2d (Step Invariance) Method

$$C_d(z) = \text{c2d}(C(s)) = \frac{z-1}{z}\mathscr{Z}\left\{ S\left(\mathscr{L}^{-1}\left\{ \frac{C(s)}{s} \right\} \right) \right\}$$

31.2 Bilinear Transformation

$$s = \frac{2}{T}\frac{z-1}{z+1} \quad C_d(z) = C\left(\frac{2}{T}\frac{z-1}{z+1} \right)$$

31.3 Pole-Zero Matching

For $C(s)$ with $d = \#\text{ poles} - \#\text{ zeros} \geq 1$, there are d infinite zeros.

$$\begin{gathered}
C(s) = K\frac{(s+b_1)(s+b_2)\dots(s+b_m)}{(s+a_1)(s+a_2)\dots(s+a_n)} \quad n \geq m \\
C_d(z) = k_d\frac{(z+1)^d(z-e^{-b_1 T})\dots(z-e^{-b_m T})}{(z-e^{-a_1 T})\dots(z-e^{-a_n T})}
\end{gathered}$$

Only add factor $(z+1)^d$ to num of $C_d(z)$ if $d = n - m > 0$. Choose K_d s.t. $C_d(1) = C(0)$